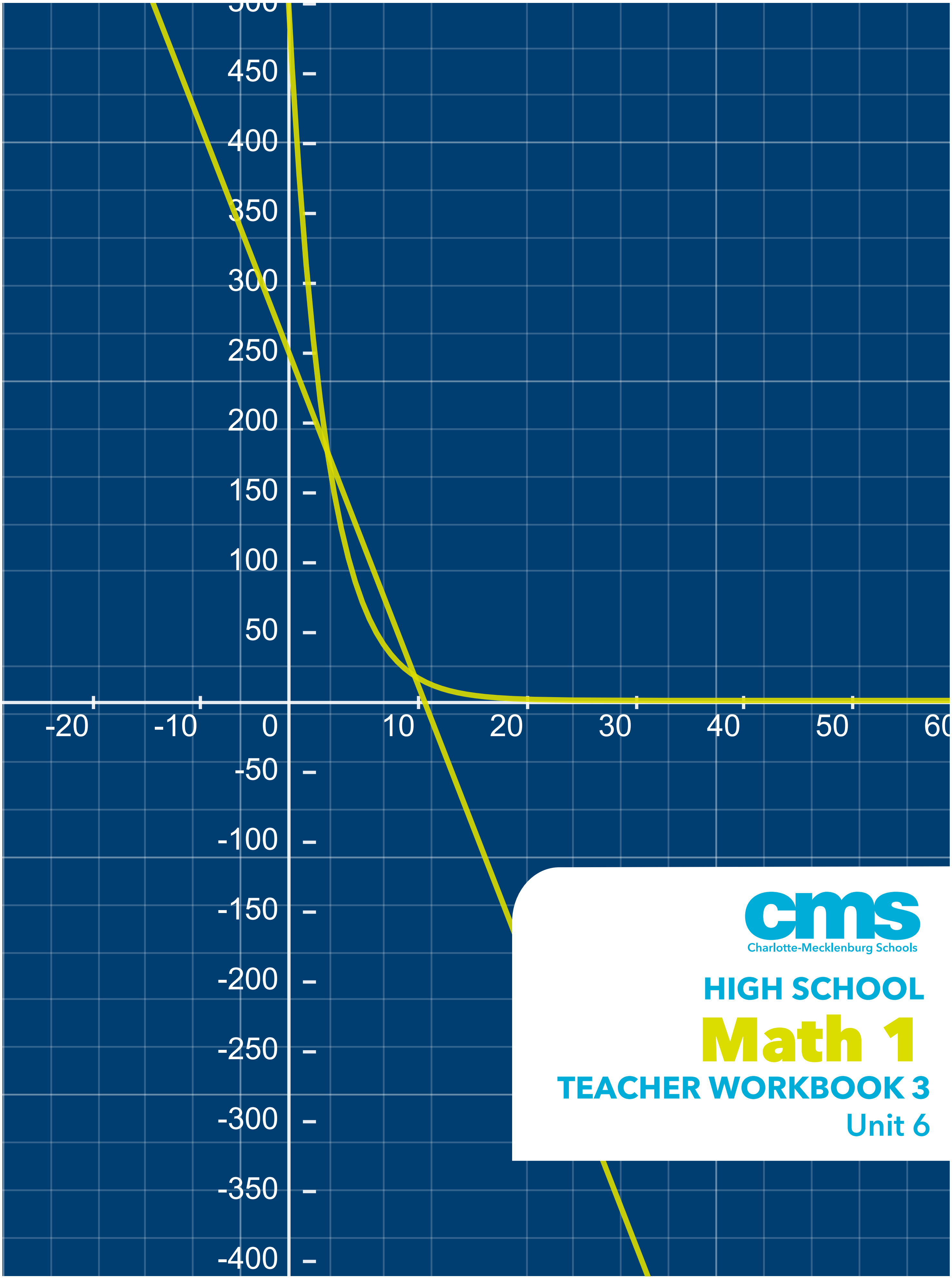



Charlotte-Mecklenburg Schools

HIGH SCHOOL

# Math 1

TEACHER WORKBOOK 3

Unit 6

## Table of Contents

## Unit 6: Exponential Functions

Before starting this unit, students are familiar with linear functions from previous units in this course and from work in grade 8. They have been formally introduced to functions and function notation and have explored the behaviors and traits of both linear and non-linear functions. Additionally, students have spent significant time graphing, interpreting graphs, and exploring how to compare the graphs of two linear functions to each other. In this unit, students are introduced to exponential functions.

In Lessons 1 and 2, students review and derive the laws of exponents. In grade 8, students learned these rules for cases where the base of the exponent was a number. Now, they apply the rules to cases where the base is a variable. Students will continue to practice applying these rules to rewrite complex expressions through practice problems across the unit. Understanding the interpretation of negative and fractional exponents is needed for students to make sense of exponential functions, where the input value does not need to be a positive integer.

In Lessons 3 and 4, students explore exponential growth patterns, formalizing these patterns as exponential growth in Lesson 5. They learn that these exponential relationships are characterized by a constant quotient over equal intervals, called the growth factor, and compare them to linear relationships, which are characterized by a constant difference over equal intervals. In Lesson 6, students see exponential relationships where a quantity decreases over time, which behave similarly but instead have a decay factor.

In Lessons 7 and 8, students continue to encounter contexts that decay exponentially. These contexts are presented verbally and with tables and graphs. Students construct equations and use them to model situations and solve problems. They investigate these exponential relationships without using function notation and language so that they can focus on gaining an appreciation for critical properties and characteristics of exponential relationships.

Lessons 9 and 10 are Checkpoint Lessons. In addition to the administration of the Check Your Readiness and small-group instruction, there are stations on reviewing percent change, open-middle style puzzles involving exponent rules, and a micro-modeling problem that involves determining which of two pizzas is the better deal. Station D deals with the difficult history of HeLa cells, introduced in Lesson 5. Students learn about the story of Henrietta Lacks, a cancer patient whose cells were cultured in the course of treatment. These cells went on to become the standard cell line in medical research and are still being used today, but at the time, the Lacks family was neither informed nor compensated. As students examine what the value of a $1,000 compensation at the time would be worth today, they are also asked to reflect on consent in the context of medical research and on how they feel the family should have been treated. This is a sensitive topic that raises issues of race, gender, and class and should be handled with care.

In Lesson 11, students begin to view exponential relationships as functions and employ the notation and terminology of functions (for example, dependent and independent variables). In Lessons 12–15, they study graphs of exponential functions both in terms of contexts they represent and abstract functions that don't represent a particular context, observing the effect of different values of $a$ and $b$ on the graph of the function $f$ represented by $f(x) = ax + b$.

Lessons 16 and 17 have students apply their understanding of percent change from grade 7 and use an exponent to express repeated increase or decrease by the same percentage. They connect the idea of percent change to the growth or decay rate of an exponential function. Lesson 17 also provides contexts in which students have more opportunity to practice solving exponential equations using technology.

In Lesson 18, students learn that the output of an increasing exponential function is eventually greater than the output of an increasing linear function for the same input. In a later unit, students are introduced to quadratic functions. At that time, students will also extend their understanding of exponential functions by considering how they relate to quadratic functions, understanding that an exponential growth function will eventually exceed both a linear and a quadratic function.

Lessons 19 and 20 are reserved for students to engage in a modeling prompt. In addition to the prompts that were included in Units 2 and 4, two additional prompts are provided here. Modeling Prompt #5 asks students to consider a fair way to divide up a bonus, and Modeling Prompt #6 investigates a concern around growing population and whether there is enough space on Earth for everyone.

Lesson 21 occurs after administering the Unit 6 assessment and includes a post-assessment activity. In this activity, students make their own decisions about how to model the population growth of three different cities, and of the world population. They may decide that linear or exponential models are appropriate, or that neither model is a good fit for the data.

Throughout the unit, the context of credit (both in terms of loans and savings) is used to:

- contextualize a percent change applied repeatedly
- make a distinction between (for example) applying a 10% increase followed by another 10% increase versus applying a 20% increase to the original amount
- strategically write and interpret expressions and relate them back to a context
- write equivalent expressions in a different way to highlight a different aspect of the situation

Note on materials: Students should have access to a calculator with an exponent button throughout the unit. Access to graphing technology is necessary for many activities in the unit, starting in Lesson 7. It is recommended that each student has their own device that can access Desmos or other graphing technology.

## Instructional Routines

Aspects of Mathematical Modeling: Lessons 7, 9 & 10, 12, 13, 19 & 20

Card Sort: Lessons 8, 17

Co-Craft Questions (MLR5): Lessons 3, 6, 15

Collect and Display (MLR2): Lessons 1, 2, 5, 6, 8, 11, 14

Compare and Connect (MLR7): Lessons 4, 7, 13, 18

Critique, Correct, Clarify (MLR3): Lessons 7, 12

Discussion Supports (MLR8): Lessons 2, 4, 5, 8, 11, 13, 14, 15, 16, 17

Graph It: Lessons 5, 7, 11, 12, 13, 14, 17, 18

Math Talk: Lesson 2

Notice and Wonder: Lessons 3, 4, 6, 7, 11, 15, 18, 19 & 20, 21

Poll the Class: Lessons 2, 3, 6, 8, 12

Round Robin: Lessons 1, 12, 19 & 20

Stronger and Clearer Each Time (MLR1): Lesson 13

Take Turns: Lessons 1, 5, 6, 8, 14, 17

Three Reads (MLR6): Lessons 3, 7, 12, 16, 18

Which One Doesn’t Belong?: Lessons 4, 15

# Lesson 1: Properties of Exponents (Part One)

## PREPARATION

| Lesson Goals | Learning Targets |
|---|---|
| • Generalize a process for multiplying exponential expressions with the same base, and justify (orally and in writing) that $x^a \cdot x^b = x^{a+b}$.<br>• Generalize a process for finding a power raised to a power, and justify (orally and in writing) that $(x^a)^b = x^{ab}$. | • I can explain and use a rule for multiplying terms with exponents that have the same base.<br>• I can explain and use a rule for raising an exponential expression to a power. |

### Lesson Narrative

In grade 8, students developed and applied the properties of integer exponents to generate equivalent numerical expressions such as $5^3 \cdot 5^8 = 5^{11}$. In Math 1, students extend the properties of integer exponents to rewrite algebraic expressions such as $x^3 \cdot x^8 = x^{11}$.

In this lesson, students revisit the definition of an exponent. They apply the definition to expand expressions with numerical bases and unknown bases. Students make use of repeated reasoning to discover the **Product of Powers Rule** and the **Power of a Power Rule** (MP8). In the next lesson, students discover the exponent rule for the quotient of a power and extend the exponent rules for when the exponents are negative. These two introductory lessons prepare students to make sense of exponential functions.

**Which Standards for Mathematical Practice do you anticipate students engaging in during this lesson? How will you support them?**

### Focus and Coherence

| Building On | Addressing |
|---|---|
| **NC.6.EE.1:** Write and evaluate numerical expressions, with and without grouping symbols, involving whole-number exponents<br><br>**NC.8.EE.1:** Develop and apply the properties of integer exponents to generate equivalent numerical expressions. | **NC.M1.N-RN.2:** Rewrite algebraic expressions with integer exponents using the properties of exponents. |

## Agenda, Materials, and Preparation

- **Warm-up** (*10 minutes*)
- **Activity 1** (*10 minutes*)
- **Activity 2** (*15 minutes*)
- **Activity 3** (*Optional, 10 minutes*)
- **Lesson Debrief** (*5 minutes*)
  - Blank visual displays for each student (possible visual display options: whiteboards/chart paper and markers, Google Slides, Jamboard)
- **Cool-down** (*5 minutes*)
  - M1.U6.L1 Cool-down (print 1 copy per student)

## LESSON

### Warm-up: Interpreting Exponents (*10 minutes*)

| **Instructional Routine:** Round Robin | |
|---|---|
| **Building On:** NC.6.EE.1 | **Building Towards:** NC.M1.N-RN.2 |

In this warm-up, students interpret the meaning of each expression by explaining, in words, the number of factors indicated by the exponent. They rewrite the expression by expanding it to show the number of factors. In later activities, students use expanded expressions to discover the exponent rules.

**Step 1**

- Ask students to arrange themselves into pairs or use visibly random grouping.
- Give students 2 minutes of quiet think time and another minute to compare with a partner using the *Round Robin* routine.

**Student Task Statement**

The expression $3^4$ means there are "4 factors of 3."

The expression can be written in the expanded form where $3^4 = 3 \cdot 3 \cdot 3 \cdot 3$.

Explain the meaning of each of the following expressions and write them in expanded form.

1. $2^4$
2. $x^3$
3. $4^3 \cdot 5^2$
4. $x^3y$
5. $(5x)^4$

**Step 2**

- Ask students to share their responses and record the expansions for all to see. If time is limited, focus the discussion on the earliest problems that many students had trouble interpreting.
- Listen for and encourage the use of the word "factor" and, in particular, phrases like "there are four factors of two." Students may want to describe the exponent as the number of times the factor is multiplied. This can be misleading as a student may interpret $2^4$ as the 2 multiplied 4 times and incorrectly expand that to show four multiplication symbols $(2 \cdot 2 \cdot 2 \cdot 2 \cdot 2)$.
- Introduce the word "power" if students have not been using it to describe an exponent. For example, $2^4$ may be read as "2 to the fourth power."

| DO THE MATH | PLANNING NOTES |
|---|---|
| | |

## Activity 1: Product of Powers (*10 minutes*)

| **Instructional Routine:** Collect and Display (MLR2) | |
|---|---|
| **Building On:** NC.8.EE.1 | **Addressing:** NC.M1.N-RN.2 |

The goal of this activity is to help students flexibly transition between different notations for powers of like bases and examine the property of multiplication of expressions with the same base. As students expand the expressions, they build an understanding of the total number of factors in the expressions. This leads to an understanding of why the exponents are added when multiplying like bases. Students conclude that expressions of the form $x^a \cdot x^b$ are equivalent to $x^{a+b}$ for values of $a$ and $b$ that are positive integers.

The third question hints at the reasoning that will extend exponent rules to include zero exponents, but students will investigate that more in the next lesson.

### Step 1

- Keep students in pairs.
- Tell students that they will be exploring patterns in the exponents to discover ways to rewrite expressions. Explain that they will write the given expression in two ways by (1) expanding the expression to show how many of each factor there are and (2) using the fewest number of exponents possible. Use the completed row to help illustrate these ways of writing the expression.
- Give students a few minutes of quiet think time to complete the table before asking students to compare with their partner. Tell students if they disagree with their partner to take turns explaining their thinking until they reach an agreement. Then ask pairs to complete questions 2 and 3 together.
- Use the *Collect and Display* routine to listen for the words and phrases students use to refer to the expressions, operations, and exponents as they work in their pairs, and scribe a variety of this language in a place where all can see. Keep this display available for updating in the next lesson.

***Advancing Student Thinking***: For the expression $3 \cdot 3^7$, students may not recognize that the first factor of 3 has an exponent of 1. Refer back to the warm-up and ask students to describe the expression in words. Students might say “3 times 3 to the seventh power.” Encourage them to interpret the expression. (The product of one factor of 3 and 7 factors of 3).

Notice students who need help writing the general rule. You might ask, “What patterns did you notice with the exponents in the table? So if the exponents are a and b, how do you write what you did with the exponents?”

## Student Task Statement

1. Complete the table to explore patterns in the exponents when multiplying expressions with the same base. You may skip a single box in the table, but if you do, be prepared to explain why you skipped it. The first row is done for you.

| Expression | Expanded | Fewest number of exponents |
|---|---|---|
| $8^2 \cdot 8^4$ | $(8 \cdot 8)(8 \cdot 8 \cdot 8 \cdot 8)$ | $8^6$ |
| a. $2^5 \cdot 2^3$ | | |
| b. $3 \cdot 3^7$ | | |
| c. $x^2 \cdot x^3$ | | |
| d. | $(m \cdot m \cdot m \cdot m \cdot m \cdot m \cdot m \cdot m \cdot m)(m \cdot m \cdot m \cdot m)$ | |
| e. $(5y^2)(4y^3)(-y)$ | | |
| f. $(r^2s^5)(r^3s^4)$ | | |

If you chose to skip one entry in the table, which entry did you skip? Why?

2. Use the patterns you found in the table to rewrite $x^a \cdot x^b$ as an equivalent expression with a single exponent, like $x^{\square}$.

3. Use your rule to write $x^a \cdot x^0$ with a single exponent. What does this tell you about the value of $x^0$?

### Step 2

- Facilitate a whole-class discussion. The purpose of the discussion is to check whether students understand why $x^a \cdot x^b = x^{a+b}$.
  - "How could you write $d^8 \cdot d^{15}$ using a single exponent without expanding all of the factors?" (There are 8 factors of $d$ and 15 factors of $d$. There are a total of 23 factors of $d$. The expression is $d^{23}$.)
  - "In general, what is a rule for multiplying two exponential expressions with the same base?" (The exponents of the two bases are added together.)
- Create and post visual displays showing the exponent rules for reference throughout the unit, with one visual display for each rule. The visual display could include an example to illustrate how the rule works, along with visual aids and use of color. Below is a sample visual display. Encourage students to color or highlight the version of this display in their workbook (in the Student Lesson Summary and Glossary section).

**Product of Powers Rule**

$$x^a \cdot x^b = x^{a+b}$$

**Why it works**

$$x^3 \cdot x^5 = \underbrace{(x \cdot x \cdot x)}_{\text{three factors of } x}\underbrace{(x \cdot x \cdot x \cdot x \cdot x)}_{\text{five factors of } x} = \underbrace{x^8}_{\text{eight factors of } x}$$

  - Tell students this is sometimes referred to as the "Product of Powers Rule." Ask students why this accurately describes the rule. (The result of multiplying is called the product, and in this rule, we are examining the product when multiplying two expressions that have powers.)
- Discuss students' responses to question 3. A more formal investigation of $x^0$ will be developed in Lesson 2.

| DO THE MATH | PLANNING NOTES |
|---|---|
| | |

## Activity 2: Power of a Power (*15 minutes*)

| **Instructional Routine**: Take Turns | |
|---|---|
| **Building On:** NC.8.EE.1 | **Addressing:** NC.M1.N-RN.2 |

As in the previous activity, students are asked to rewrite expressions in different ways, this time to examine the property of applying a power to a power. As students expand the expressions, they begin to recognize that the expressions are grouped in such a way that multiplication can be used to find the total number of factors. Students conclude that expressions such as $(x^a)^b$ are equivalent to $x^{ab}$ for values of $a$ and $b$ that are positive integers.

### Step 1

- Display the expressions $x^3$ and $(x^2)^3$.
- Provide students a minute to consider how the expressions are similar and how they are different.
- Ask students to share. (They both have an expression that is being raised to the third power. The first has the expression $x$ and the second has the expression $x^2$.)
- Tell students that they will be exploring patterns in the exponents to discover ways to rewrite expressions where there is a power to a power such as $(x^2)^3$.
- Refer to the first row in the table. Explain that students may choose to do only a partial expanded expression such as $3^2 \cdot 3^2 \cdot 3^2 \cdot 3^2 \cdot 3^2$ instead of a full expanded expression such as $(3 \cdot 3)(3 \cdot 3)(3 \cdot 3)(3 \cdot 3)(3 \cdot 3)$ and can then apply the product of a power rule to write the last expression, $3^{10}$.

Give students time to work in pairs to complete the task. Instruct students to *Take Turns* when completing each row of the table, reversing the order each time. This will ensure active independent engagement with each expression. After completing the table, pairs can work collaboratively to complete questions 2 and 3.

***Advancing Student Thinking***: In part d, students may forget that the number 6 is part of the exponential expression, and expand in such a way that the simplified form becomes $6n^{12}$. Remind these students that the exponent of 3 means there are 3 factors of $6n^4$. In part e, students may not know how to handle the negative sign, or, if they do not use parentheses in their expansion, think that subtraction is involved. It may help these students not only to write the expanded expression but also to rearrange the terms so that the two factors of -3 are written first, then the two factors of $r^3$, then the two factors of $s^4$. It may help to remind students as well that the product of two negative numbers is positive.

## Student Task Statement

1. Complete the table to explore patterns in the exponents when applying a power to a power. The first row is done for you.

| Expression | Expanded | Fewest number of exponents |
|---|---|---|
| $(3^2)^5$ | $3^2 \cdot 3^2 \cdot 3^2 \cdot 3^2 \cdot 3^2 = (3 \cdot 3)(3 \cdot 3)(3 \cdot 3)(3 \cdot 3)(3 \cdot 3)$ | $3^{10}$ |
| a. $(2^3)^4$ | | |
| b. | $x^4 \cdot x^4 = (x \cdot x \cdot x \cdot x)(x \cdot x \cdot x \cdot x)$ | |
| c. $(xy^2)^3$ | | |
| d. $(6n^4)^3$ | | |
| e. $(-3r^3s^4)^2$ | | |

2. Use the patterns you found in the table to rewrite $(x^a)^b$ as an equivalent expression with a single exponent, like $x^{\square}$.

3. Jada rewrote the expression $(3x^5)^2(2x^6)$. Examine her work and decide if you agree or not.
   - If you agree, explain why each step is correct.
   - If you disagree, explain the error and provide the correct equivalent expression.

$$\begin{aligned}(3x^5)^2(2x^6) &= 3^2 \cdot x^{10} \cdot 2 \cdot x^6 \\ &= 6 \cdot 2 \cdot x^{10} \cdot x^6 \\ &= 12x^{16}\end{aligned}$$

### Step 2

- Select students who can explain the patterns they noticed to share in a whole-class discussion.
- Ask: "In general, what is a rule to rewrite an expression that has a power to a power as an expression with a single exponent?" (The exponents are multiplied.)
- Create a visual display for the rule $(x^a)^b = x^{ab}$ to display for all to see throughout the unit. For an example of how the rule works, consider showing $(x^2)^3 = (x \cdot x)(x \cdot x)(x \cdot x) = x^6$ using colors or other visual aids to highlight that the result is $x^6$ because there are three groups of $x^2$ as seen below. Encourage students to color or highlight the version of this display in their workbook (in the Student Lesson Summary and Glossary section).

**Power of a Power Rule**

$$(x^a)^b = x^{ab}$$

**Why it works**

$$(x^2)^3 = \underbrace{(x \cdot x)(x \cdot x)(x \cdot x)}_{\text{three groups of two factors of } x} = \underbrace{x^6}_{\text{six factors of } x}$$

  - Tell students this is sometimes referred to as the "Power of a Power Rule." Ask students why this accurately describes the rule. (In this rule, we are taking a base that has a power and raising it to another power.)

- Display Jada's work from question 3 and ask students to share their responses. (There is an error in the second step: $3^2$ is 9, not 6.) Also answer any questions students may have about the correct steps in Jada's work.

| DO THE MATH | PLANNING NOTES |
|---|---|
| | |

## Activity 3: More Than One Rule (*Optional, 10 minutes*)

| **Building On:** NC.8.EE.1 | **Addressing:** NC.M1.N-RN.2 |
|---|---|

In this optional activity, students rewrite expressions where more than one rule will be applied. As students build towards procedural fluency, they may need to expand the expression in a way similar to the reasoning used in developing the rules.

### Step 1

- Keep students in pairs.
- Give students a few minutes of quiet think time to complete the four problems before asking students to compare with their partner. Tell students if they disagree with their partner to take turns explaining their thinking until they reach an agreement.

**Monitoring Tip:** Look for students who use different strategies to rewrite the expressions in the following ways:

- expanding the expressions completely, such as $(x \cdot x \cdot x \cdot x \cdot x)(x \cdot x \cdot x \cdot x \cdot x)(x \cdot x \cdot x)$
- expanding the expressions partially, such as $(x^5)(x^5)(x^3)$, and then applying a rule
- applying the Power of a Power Rule $(x^{10})(x^3)$ and then the Product of Powers Rule $(x^{15})$

### Student Task Statement

Rewrite each expression using the fewest possible exponents. Show or explain your reasoning.

1. $(x^5)^2(x^3)$
2. $(6y^3)^2(y^4)$
3. $(rs^2)(r^6)^2(s)$
4. $(3m)^2(2m^4n^7)$

### Step 2

- Ask previously selected students to share how they rewrote the expressions in the order listed in the Monitoring Tip.
- Display the different strategies for all to see.
- Facilitate a whole-class discussion. If time allows, discuss all four problems. If time is limited, select problems that will bring out important understandings. Ask students questions such as:
  - "How do the different strategies show the number of factors?" (Possible response for problem 1: When expanded completely you see the thirteen factors of $x$. When partially expanded, you can see two groups of five factors of $x$, which are ten factors of $x$. There are also an additional three factors of $x$ for a total of

thirteen factors of $x$. In the original expression, the power of a power means two groups of five factors without having to write both. There are still the three additional factors of $x$.)

- “Is there a specific order you should use when applying the rules?” (Look for and apply the Power of a Power Rule by multiplying the exponents before using the Product of Powers Rule to add the exponents.)
- “Why might a student rewrite $(6y^3)^2$ as $12x^6$, and what would you tell the student to help them correct their expression?” (The student multiplied $6 \cdot 2$ instead of raising 6 to the power of 2. I would tell the student that the power of 2 means there are two factors of 6 and $6 \cdot 6 = 36$.)

| DO THE MATH | PLANNING NOTES |
|---|---|
| | |

## Lesson Debrief (*5 minutes*)

The purpose of this lesson is to remind students of two exponent rules they learned in middle school: the Product of Powers Rule and the Power of a Power Rule. Students use the fact that exponents are repeated multiplication to derive and deepen their understanding of these rules.

If necessary, choose what questions to focus the discussion on.

The purpose of this discussion is to help students become flexible in applying the exponent rules to write equivalent exponential expressions. Students have the tools to answer the first two questions even though they will require some new thinking.

- Display the two rules developed in this lesson for all to see.
- Provide students with visual display tools (whiteboards/chart paper and markers, Jamboard, etc.).
- Display each question. Give students time to respond and show their work.
- Ask students to explain their reasoning.
  - Find the value of $b$: $x^7 \cdot x^b = x^{10}$. ($b = 3$ because $7 + 3 = 10$.)
  - Find the value of $a$: $(x^a)^4 = x^{24}$. ($b = 4$ because $6 \cdot 4 = 24$.)
  - Rewrite using the fewest exponents possible: $3x^7 \cdot 8x^4$. (The expression is equivalent to $24x^{11}$.)
  - Rewrite using the fewest exponents possible: $(-3r^5)^2$. (The expression is equivalent to $9r^{10}$.)

**PLANNING NOTES**

## Student Lesson Summary and Glossary

Exponents are used to express repeated multiplication. For example, $x^5$ means there are five factors of $x$ and can be expanded into the equivalent form of $x \cdot x \cdot x \cdot x \cdot x$.

Expressions that include exponents can be written in different ways by using the properties of exponents.

One of these properties is called the **Product of Powers Rule**. This is used when multiplying exponential expressions with the same base.

**Product of Powers Rule**

$$x^a \cdot x^b = x^{a+b}$$

**Why it works**

$$x^3 \cdot x^5 = \underbrace{(x \cdot x \cdot x)}_{\text{three factors of } x}\underbrace{(x \cdot x \cdot x \cdot x \cdot x)}_{\text{five factors of } x} = \underbrace{x^8}_{\text{eight factors of } x}$$

Another of these properties is called the **Power of a Power Rule**. This is used when applying a power to an expression that also has a power.

**Power of a Power Rule**

$$(x^a)^b = x^{ab}$$

**Why it works**

$$(x^2)^3 = \underbrace{(x \cdot x)(x \cdot x)(x \cdot x)}_{\text{three groups of two factors of } x} = \underbrace{x^6}_{\text{six factors of } x}$$

For some expressions, both properties are needed. For example, in the expression $(4x^3)^2(x^5)$:

$(4x^3)^2(x^5)$ — The expression $4x^3$ is being raised to a power of $2$, so multiply each exponent by $2$.(The $4$ has an exponent of $1$ and can be written as $4^1$: $(4^1x^3)^2$)

$= (4^2x^6)(x^5)$ — The expressions $x^6$ and $x^5$ have the same base and are being multiplied, so add the exponents. Rewrite $4^2$ as $16$.

$= 16x^{11}$

**Product of Powers Rule**: When multiplying two exponential expressions that have the same base, add the exponents: $x^a \cdot x^b = x^{a+b}$

**Power of a Power Rule**: When raising an exponential expression to a power, multiply the powers: $(x^a)^b = x^{ab}$

## Cool-down: Powers, Part One (*5 minutes*)

**Addressing:** NC.M1.N-RN.2

**Cool-down Guidance:** More Chances
Students will continue to practice rewriting exponential expressions in Lesson 2 and throughout the unit in practice problems.

### Cool-down

Rewrite each expression using the fewest number of exponents.

1. $y^6(y^3)$
2. $(4rs^7)^3$
3. $(10n)(n^3)^2$

**Student Reflection:**

How are you feeling about starting this new unit? What are you most excited about? What are you most nervous about?

### DO THE MATH

### INDIVIDUAL STUDENT DATA

### SUMMARY DATA

## NEXT STEPS

## TEACHER REFLECTION

**What went well in the lesson? What would you do differently next time? What happened today that will influence the planning of future lessons?**

**In the next lesson, students will learn the Quotient of Powers Rule. What do you notice in their work from today's lesson that you might leverage in the next lesson?**

## Practice Problems

1. Rewrite the expression using the fewest possible exponents.

   a. $x^3 \cdot x^{10}$

   b. $(s^6)^5$

   c. $(4m^2)(-m^8)(m)$

   d. $(2d^5f)^3(d^4f^7)$

2. Find the value of each unknown exponent that makes the equation true.

   a. Find the value of $b$: $3y^7 \cdot 2y^b = 6y^{12}$

   b. Find the value of $b$: $(4x^6)^b = 64x^{18}$

   c. Find the values of $b$ and $c$: $(r^9s^3)(7r^2s)^b = 49r^cs^5$

3. Select the **four** expressions that are equivalent to $-2x^5(4x^3)^2$

   a. $-2x^5(4x^3)(4x^3)$
   b. $(-2 \cdot 4)(x^5 \cdot x^3 \cdot x^2)$
   c. $-2x^5(16x^6)$
   d. $(-2 \cdot x \cdot x \cdot x \cdot x \cdot x)(4 \cdot x \cdot x \cdot x)(4 \cdot x \cdot x \cdot x)$
   e. $-8x^{10}$
   f. $-32x^{11}$
   g. $-512x^{11}$

4. Which expression is **equal to** $(6x^5y)^3(xy^4)$?

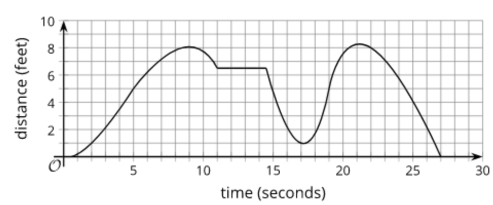
   a. $6x^9y^7$
   b. $6x^{16}y^7$
   c. $18x^{16}y^7$
   d. $216x^{16}y^7$

5. A tennis ball is dropped from an initial height of 30 feet. It bounces five times, with each bounce height being about $\frac{2}{3}$ of the height of the previous bounce.

   Sketch a graph that models the height of the ball over time. Be sure to label the axes.


   (From Unit 5)

6. This graph represents Andre's distance from his bicycle as he walks in a park.




   a. For which intervals of time is the value of the function decreasing?

   b. For which intervals is it increasing?

   c. Describe what Andre is doing during the time when the value of the function is increasing.

   (From Unit 5)

7. (*Technology required.*) A survey wanted to determine if there was a relationship between the number of joggers who used a local park for exercise and the temperature outside. The data in the table display their findings.

| Temperature in Fahrenheit, $x$ | Number of joggers, $y$ |
|---|---|
| 15 | 4 |
| 30 | 8 |
| 30 | 8 |
| 41 | 4 |
| 42 | 16 |
| 49 | 20 |
| 49 | 14 |
| 55 | 16 |
| 66 | 34 |
| 72 | 44 |
| 85 | 40 |
| 94 | 15 |

Use graphing technology to create a scatter plot of the data.

a. Is a linear model appropriate for this data? Explain your reasoning.

b. If the data seems appropriate, create the line of best fit. Round to two decimal places.

c. What is the slope of the line of best fit and what does it mean in this context? Is it realistic?

d. What is the $y$-intercept of the line of best fit and what does it mean in this context? Is it realistic?

(From Unit 4)

8. Select **all** equations that can result from adding these two equations or subtracting one from the other.

$$\begin{cases} x+y=12 \\ 3x-5y=4 \end{cases}$$

a. $-2x-4y=8$
b. $-2x+6y=8$
c. $4x-4y=16$
d. $4x+4y=16$
e. $2x-6y=-8$
f. $5x-4y=28$

(From Unit 3)

9. Solve each system of equations without graphing.

a. $\begin{cases} 2x+3y=5 \\ 2x+4y=9 \end{cases}$

b. $\begin{cases} \frac{2}{3}x+y=\frac{7}{3} \\ \frac{2}{3}x-y=1 \end{cases}$

(From Unit 3)

10. Select **all** the expressions that equal $2^8 \cdot 2^4$.

a. $2^4$
b. $2^{12}$
c. $4^6$
d. $2^{32}$
e. $4^{32}$

(Addressing NC.6.EE.1)

# Lesson 2: Properties of Exponents (Part Two)

## PREPARATION

| Lesson Goals | Learning Targets |
|---|---|
| • Generalize a process for dividing exponential expressions and justify (orally and in writing) that $\frac{x^a}{x^b} = x^{a-b}$.<br>• Use exponent rules to justify (orally) that $x^0 = 1$.<br>• Use exponent rules to explain why $x^{-a} = \frac{1}{x^a}$. | • I can explain and use a rule for dividing exponential expressions.<br>• I can evaluate $x^0$ and explain why it makes sense.<br>• I know what it means if an expression is raised to a negative power. |

### Lesson Narrative

This is the second lesson on exponent properties. In grade 8, students developed and applied the properties to numerical expressions. In Math 1, these properties are extended to algebraic expressions. In the first activity, students continue to use repeated reasoning to discover the **Quotient of Powers Rule** $\frac{x^a}{x^b} = x^{a-b}$ (MP8). Students learn that they can separate as many factors as possible from the numerator and denominator to create a fraction that is equivalent to 1. This allows them to see that the number of factors remaining should be the $a$ factors in the numerator minus the $b$ factors in the denominator. For example, $\frac{x^5}{x^3} = \frac{x \cdot x \cdot x}{x \cdot x \cdot x} \cdot x^2 = 1 \cdot x^2$, which is the same as $x^{5-3} = x^2$.

In the previous lesson, students reasoned that $x^0 = 1$ based on the Products of Powers Rule. In this lesson, they will use the Quotient of Powers Rule and the case where $a = b$ as another way to make sense of why that is true (MP3).

Lastly, students will revisit negative exponents. Students will use two strategies for exploring $\frac{x^a}{x^b}$ when $a < b$. First, when the expressions are expanded, students recognize that there are more factors in the denominator than in the numerator. For example, $\frac{x^4}{x^5} = \frac{x \cdot x \cdot x \cdot x}{x \cdot x \cdot x \cdot x} \cdot \frac{1}{x} = 1 \cdot \frac{1}{x} = \frac{1}{x}$. Next, when using the Quotient of Powers Rule, the expression has a negative exponent such as $\frac{x^4}{x^5} = x^{4-5} = x^{-1}$. Thus, students conclude that since $\frac{x^4}{x^5} = \frac{1}{x}$ and $\frac{x^4}{x^5} = x^{-1}$ then $x^{-1} = \frac{1}{x}$. This is summarized as $x^{-a} = \frac{1}{x^a}$.

**What teaching strategies will you be focusing on during this lesson?**

## Focus and Coherence

| Building On | Addressing |
| --- | --- |
| **NC.4.NF.1:** Explain why a fraction is equivalent to another fraction by using area and length fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size.<br><br>**NC.8.EE.1:** Develop and apply the properties of integer exponents to generate equivalent numerical expressions. | **NC.M1.N-RN.2:** Rewrite algebraic expressions with integer exponents using the properties of exponents. |

## Agenda, Materials, and Preparation

- **Bridge** (*Optional, 5 minutes*)
- **Warm-up** (*5 minutes*)
- **Activity 1** (*10 minutes*)
- **Activity 2** (*15 minutes*)
  - Blank visual displays for each student (possible visual display options: whiteboards and markers, chart paper, Google Slides, Jamboard)
- **Lesson Debrief** (*5 minutes*)
- **Cool-down** (*5 minutes*)
  - M1.U6.L2 Cool-down (print 1 copy per student)

## LESSON

### Bridge (*Optional, 5 minutes*)

**Building On:** NC.4.NF.1

In the lesson, students divide exponential expressions with numerical coefficients. For example, they rewrite the expression $\frac{6xy^3}{15y^2}$, which includes writing $\frac{6}{15}$ as $\frac{2}{5}$. Students will also be working with fractions that are equivalent to 1. The purpose of the bridge is for students to revisit equivalent fractions. Use student responses to illustrate the concept that $\frac{6}{15} = \frac{3 \cdot 2}{3 \cdot 5} = 1 \cdot \frac{2}{5} = \frac{2}{5}$.

#### Student Task Statement

Name two fractions that are equivalent to $\frac{2}{5}$. Explain or show your reasoning.[1]

| DO THE MATH | PLANNING NOTES |
| --- | --- |
| | |

[1] Adapted from IM K–5, Grade 4, Unit 2, Lesson 7 https://curriculum.illustrativemathematics.org/K5/index.html, copyright 2019 by Illustrative Mathematics. Licensed under the Creative Commons Attribution 4.0 license https://creativecommons.org/licenses/by/4.0/.

## Warm-up: A Value of One (*5 minutes*)

| **Instructional Routines:** Math Talk; Discussion Supports (MLR8) - Responsive Strategy | |
|---|---|
| **Building On:** NC.8.EE.1 | **Addressing:** NC.M1.N-RN.2 |

This *Math Talk* encourages students to reason with fractions that are equal to 1. This is an important concept that helps students make sense of the quotient of powers rule explored later in this lesson. Students are given four equations and are asked to solve each equation. Students should look for and make use of the structure (MP7) by recognizing the value of $x$ must be the same as either the number in the numerator or in the denominator for the fraction to equal 1. If there are problems for which students do not use this strategy, be sure to bring it up.

**MATH TALK**

***What Is This Routine?*** In these warm-ups, one problem is displayed at a time. Students are given a few moments to quietly think and give a signal when they have an answer and a strategy. The teacher selects students to share different strategies for each problem, asking, "Who thought about it a different way?" Their explanations are recorded for all to see. Students might be pressed to provide more details about why they decided to approach a problem a certain way. It may not be possible to share every possible strategy in the given time—the teacher may only gather two or three distinctive strategies per problem. Problems are purposefully chosen to elicit different approaches, often in a way that builds from one problem to the next.

***Why This Routine?*** A *Math Talk* builds fluency by encouraging students to think about the numbers, shapes, or algebraic expressions and rely on what they know about structure, patterns, and properties of operations to mentally solve a problem. While participating in these activities, there is a natural need for students to be precise in their word choice and use of language (MP6). Additionally, a Math Talk often provides opportunities to notice and make use of structure (MP7).

### Step 1

- Display one problem at a time.
- Give students 1–2 minutes of quiet think time per problem and ask them to give a signal when they have their answers and are ready to explain their strategies.

### Student Task Statement

Solve each equation mentally.

1. $\frac{12}{x} = 1$
2. $\frac{x}{9} = 1$
3. $\frac{x}{4} \cdot 5 = 5$
4. $\frac{3 \cdot 7 \cdot 11}{3 \cdot x} = 7$

### Step 2

- Facilitate a whole-class discussion by asking students to share their strategies for each problem. Record and display their responses for all to see. To involve more students in the conversation, consider asking:
  - "Who can restate ____'s reasoning in a different way?"
  - "Did anyone have the same strategy but would explain it differently?"
  - "Did anyone solve the problem in a different way?"
  - "Does anyone want to add on to ____'s strategy?"
  - "Do you agree or disagree? Why?"

**RESPONSIVE STRATEGY**

Display sentence frames to support students when they explain their strategy. For example, "First, I ___ because..." or "I noticed ___ so I ...." Some students may benefit from the opportunity to rehearse what they will say with a partner before they share with the whole class.

Discussion Supports (MLR8)

| DO THE MATH | PLANNING NOTES |
| --- | --- |
| | |

## Activity 1: Quotient of Powers (*10 minutes*)

| **Instructional Routine:** Collect and Display (MLR2) | |
| --- | --- |
| **Building On:** NC.8.EE.1 | **Addressing:** NC.M1.N-RN.2 |

The goal of this activity is for students to use the expanded expressions in the numerator and denominator to identify the fraction that equals 1 and compute the number of any remaining factors.

### Step 1

- Before students begin working, discuss the "expanded" column for the expressions $\frac{2^{10}}{2^3}$ and $\frac{8x^5}{6x^3}$.
  - Ask students to review the steps in the examples in the first and second rows of the table and think about the justifications for each step in the "expanded" column.
  - Call on a volunteer to describe the rationale for each step.
  - Call on another student to revoice what they heard the first volunteer describe, trying to elevate the voice of a student who does not typically volunteer. If they are stuck, encourage the first volunteer to repeat their explanation so the student can revoice.
  - It is important for students to understand that the "expanded" column shows each expression expanded into factors and a certain number of factors in the numerator and denominator being grouped because their quotient is 1.
- Ask students to arrange themselves into pairs or use visibly random grouping.
- Give students a few minutes of quiet think time to complete the table before asking students to compare with their partner. Tell students if they disagree with their partner to take turns explaining their thinking until they reach an agreement. Then ask pairs to complete questions 2 and 3 together.
  - Use the *Collect and Display* routine to listen for the words and phrases students use as they work in their pairs, and add these to the display from the previous lesson. If students are using new language to refer to the expressions, operations, and exponents today, annotate the display to indicate additional and refined words and phrases.

## Student Task Statement

1. Complete the table to explore patterns in the exponents when dividing expressions with exponents. You may skip a single box in the table, but if you do, be prepared to explain why you skipped it. The first two rows are done for you.

| Expression | Expanded | Fewest number of exponents |
|---|---|---|
| $\frac{2^{10}}{2^3}$ | $\frac{2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2}{2\cdot2\cdot2} = \frac{2\cdot2\cdot2}{2\cdot2\cdot2} \cdot 2^7 = 1 \cdot 2^7$ | $2^7$ |
| $\frac{8x^5}{6x^3}$ | $\frac{2\cdot4\cdot x\cdot x\cdot x\cdot x\cdot x}{2\cdot3\cdot x\cdot x\cdot x} = \frac{2\cdot x\cdot x\cdot x}{2\cdot x\cdot x\cdot x} \cdot \frac{4}{3} \cdot x \cdot x = 1 \cdot \frac{4}{3}x^2$ | $\frac{4}{3}x^2$ |
| a. $\frac{15x^6}{5x^2}$ | | |
| b. | $\frac{2\cdot5\cdot r\cdot r\cdot r\cdot r\cdot r}{2\cdot r\cdot r\cdot r} = \frac{2\cdot r\cdot r\cdot r}{2\cdot r\cdot r\cdot r} \cdot 5r^2 = 1 \cdot 5r^2$ | |
| c. $\frac{-8m^4n^7}{2m^3n}$ | | |
| d. $\frac{6xy^3}{15y^2}$ | | |

If you chose to skip one entry in the table, which entry did you skip? Why?

2. Use the patterns you found in the table to write $\frac{x^a}{x^b}$ as an equivalent expression with a single exponent, like $x^{\square}$.

3. Use your rule to write each of the following with a single exponent, like $x^{\square}$.

| Expression | Expanded | Expression with a single exponent |
|---|---|---|
| a. $\frac{x^3}{x^3}$ | $\frac{x\cdot x\cdot x}{x\cdot x\cdot x} = 1$ | |
| b. $\frac{r^4}{r^4}$ | $\frac{r\cdot r\cdot r\cdot r}{r\cdot r\cdot r\cdot r} = 1$ | |
| c. $\frac{m^5}{m^5}$ | $\frac{m\cdot m\cdot m\cdot m\cdot m}{m\cdot m\cdot m\cdot m\cdot m} = 1$ | |

.

4. What does this tell you about the value of $x^0$? Explain your reasoning.

### Step 2

- Facilitate a whole-class discussion. The purpose of the discussion is to check whether students understand why $\frac{x^a}{x^b} = x^{a-b}$. Refer to the language collected and displayed as relevant, and invite students to clarify and revise their words and phrases as they discuss. Here are some questions for discussion:
  - "How could you write $\frac{d^{22}}{d^{10}}$ using a single exponent without expanding all of the factors?" (There are 22 factors of $d$ in the numerator and 10 factors of $d$ in the denominator. When you group 10 of them in both the numerator and denominator, you are left with a fraction that is equal to 1, multiplied by the 12 leftover factors of $d$. The expression is $d^{12}$.)

- "In general, what is a rule for dividing two exponential expressions with the same base?" (The exponent in the denominator is subtracted from the exponent in the numerator.)

- Continue creating and displaying exponent rules. Continue encouraging students to color or highlight the version of this display in their workbook (in the Student Lesson Summary and Glossary section).
- Here is a sample visual display for the Quotient of Powers Rule:

| Quotient of Powers Rule | Why it works |
|---|---|
| $\frac{x^a}{x^b} = x^{a-b}$ | $\frac{x^5}{x^3} = \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x} = \frac{x \cdot x \cdot x}{x \cdot x \cdot x} \cdot x^2 = 1 \cdot x^2 = x^{5-3}$<br>three factors of $x$ in the numerator and denominator<br>5-3=2 factors of $x$ |

- Ask students to share their thinking about what $x^0$ means.
- Display the Zero Exponent Rule.

| Zero Exponent Rule | Why it works |
|---|---|
| $x^0 = 1$ | Using the expanded form: $\frac{x^3}{x^3} = \frac{x \cdot x \cdot x}{x \cdot x \cdot x} = 1$<br>Using the Quotient of Powers Rule: $\frac{x^3}{x^3} = x^{3-3} = x^0$<br>Since $\frac{x^3}{x^3} = 1$ and $\frac{x^3}{x^3} = x^0$ then $x^0 = 1$ |

| DO THE MATH | PLANNING NOTES |
|---|---|
| | |

## Activity 2: Negative Exponents (*15 minutes*)

| **Instructional Routine**: Poll the Class | |
|---|---|
| **Building On:** NC.8.EE.1 | **Addressing:** NC.M1.N-RN.2 |

In this activity, students rewrite a quotient by using the expanded form and by using the Quotient of Powers Rule. The application of the rule results in a negative exponent. Students reason that the two results are equivalent. They look for and make use of structure (MP7) to see that they can rewrite expressions with negative exponents as an equivalent expression with positive exponents.

Sometimes in mathematics, extending existing concepts to areas outside of the original definition leads to new insights and new ways of thinking. Students adapt the definition of an exponent to interpret the negative sign as the reciprocal of the factor. Lastly, students apply the exponent rules they know, now including negative exponents.

### Step 1

- Provide each student with a whiteboard and marker. Tell students they will use these materials to show their thinking during the discussion.
- Keep students in pairs.
- Display the expression: $\frac{x^5}{x^7}$.
- Ask one student in the pair to rewrite the expression using the Quotient of Powers Rule and the other student in the pair to use the expanded form, as in the last activity.
- Give students a minute to rewrite the expression with their given strategy and another minute to compare results with their partner.
- Ask pairs to share their results and record the results for all to see. (The expanded form shows that $\frac{x^5}{x^7} = \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x \cdot x \cdot x} \cdot \frac{1}{x^2} = 1 \cdot \frac{1}{x^2} = \frac{1}{x^2}$ and the Quotient of Powers Rule shows that $\frac{x^5}{x^7} = x^{5-7} = x^{-2}$.)
- Facilitate a class discussion. Possible questions include:
  - "Why is there a negative exponent?" (The larger exponent is in the denominator. When you subtract a larger number from a smaller number, you get a negative number.)
  - "How did the expanded form show that there were more factors of $x$ in the denominator?" (When the factors of $x$ were grouped, there were five factors of $x$ in the numerator and denominator with two more factors of $x$ in the denominator. This was written as $\frac{1}{x^2}$.) As students describe, annotate their thinking on the record of the expanded form.
  - "Using the results of the two strategies, what can we conclude about $x^{-2}$ and $\frac{1}{x^2}$?" (Since $\frac{x^5}{x^7} = x^{-2}$ and $\frac{x^5}{x^7} = \frac{1}{x^2}$, then $x^{-2} = \frac{1}{x^2}$.)
  - "Describe how the two expressions $x^{-2}$ and $\frac{1}{x^2}$ are different." (The exponent in one of the expressions is a negative two, and in the other expression, it is a positive two. The term with the exponent is in the numerator for one of the expressions and in the denominator in the other expression.)
- Tell students: "Let's look at interpreting the negative exponent. Consider the expression $3^{-2}$. Interpreting this exponent as -2 factors of 3 does not make sense." Ask students to use their whiteboards to record their thinking for the following questions.
  - "How can we write the expression with a positive exponent?" (Write it with $3^2$ in the denominator to get $\frac{1}{3^2}$).
  - "How might you interpret this new expression?" (There are two factors of 3 in the denominator.)
  - Describe how to interpret the negative exponent and use it to write expressions with positive exponents. (The negative exponent means that the factor should be written as its reciprocal with a positive exponent.)
- Students might not use the vocabulary of "reciprocal." The reciprocal of $3^2$ is $\frac{1}{3^2}$. Ask "Where have you heard or used the word reciprocal before?" (The slope of a perpendicular line was the opposite reciprocal.) Use this as an opportunity to revisit the word and illustrate with a few examples. The reciprocal of 2 is $\frac{1}{2}$. The reciprocal of $\frac{2}{5}$ is $\frac{5}{2}$. The reciprocal of $\frac{1}{6}$ is 6.
- Display the Negative Exponent Rule as seen below. Again, encourage students to color or highlight the version of this display in their workbook (in the Student Lesson Summary and Glossary section).

| Negative Exponent Rule | Why it works |
|---|---|
| $x^{-a} = \frac{1}{x^a}$ | Using the expanded form<br>$\frac{x^3}{x^5} = \frac{x \cdot x \cdot x}{x \cdot x \cdot x \cdot x \cdot x} = \frac{x \cdot x \cdot x}{x \cdot x \cdot x} \cdot \frac{1}{x^2} = 1 \cdot \frac{1}{x^2} = \frac{1}{x^2}$<br>Using the Quotient of Powers Rule<br>$\frac{x^3}{x^5} = x^{3-5} = x^{-2}$<br>Since $\frac{x^3}{x^5} = \frac{1}{x^2}$ and $\frac{x^3}{x^5} = x^{-2}$ then $x^{-2} = \frac{1}{x^2}$ |

- Display each of the following expressions one at a time.
  - $4x^{-3}$
  - $(2m)^{-6}$
  - $\frac{1}{r^{-2}}$
- Ask students to write the expression using positive exponents and, when prompted, show their board. Ask students to explain their reasoning.
  - $4x^{-3}$ (The reciprocal of $x^3$ is $\frac{1}{x^3}$ so $4x^{-3} = \frac{4}{x^3}$ .)
  - $(2m)^{-6}$ (The reciprocal of $2m$ is $\frac{1}{2m}$ so $(2m)^{-6} = \frac{1}{(2m)^6}$.)
  - $\frac{1}{r^{-2}}$ (The reciprocal of $\frac{1}{r^2}$ is $r^2$ so $\frac{1}{r^{-2}} = r^2$.)

## Step 2

- Tell students they will work on rewriting algebraic expressions by applying the rules developed in both this lesson and the last lesson. Refer to the rules displayed.
- Explain that while they might have negative exponents when working with the expressions, the final expression should be written with only positive exponents.
- Give students a few minutes of quiet think time and then time to compare with their partner. Tell students if they disagree with their partner to take turns explaining their thinking until they reach an agreement.

M **Monitoring Tip:** As students work, monitor and look for the order students apply the exponent rules to rewrite expressions 4 and 5. Select students to show the different approaches (possibly two different approaches per expression). Let students know that they may be asked to share later. Include at least one student who does not typically volunteer.

### Student Task Statement

Rewrite the expressions using the rules of exponents. The final expression should include only positive exponents.

1. $x^{-7} \cdot x^3$
2. $(2rs^2)^{-3}$
3. $\frac{8m^6}{4m^9}$
4. $(3x^2y)^{-1}(6x^3y^4)$
5. $\frac{(r^4s^{-2})^3}{rs^4}$

### Step 3

- Ask student volunteers to share responses to expressions 1–3. Invite students to refer to the visual displays of language and the exponent rules as they share.
- Ask selected students to present their strategies for expressions 4 and 5. After the strategy is presented, *Poll the Class* for others who approached it the same way.

| DO THE MATH | PLANNING NOTES |
|---|---|
| | |

## Lesson Debrief (*5 minutes*)

The purpose of this lesson is to remind students of the Quotient of Powers Rule and introduce the interpretation of negative exponents. Emphasis is not only on the rules themselves but also on understanding why those rules are true, so that students have tools to re-derive those rules in the future.

Choose what questions to focus the discussion on, whether students should first have an opportunity to reflect in their workbooks or talk through these with a partner, and what questions will be prioritized in the full class discussion.

Use the following questions in order to check that students can explain why the exponents are subtracted when rewriting a quotient of powers and how to write expressions that have negative exponents as expressions with positive exponents.

- "How can you write $\frac{x^{30}}{x^{22}}$ using a single exponent?"

  $(\frac{x^{30}}{x^{22}} = x^{30-22} = x^8)$

- "How can you write $x^{-3}$ using a positive exponent?" $(x^{-3} = \frac{1}{x^3})$
- "How would you write $\frac{1}{x^{-2}}$ using a positive exponent?" $(\frac{1}{x^{-2}} = x^2)$

**PLANNING NOTES**

## Student Lesson Summary and Glossary

The **Quotient of Powers Rule** is used when dividing expressions with exponents.

**Quotient of Powers Rule**

$$\frac{x^a}{x^b} = x^{a-b}$$

**Why it works**

$$\frac{x^5}{x^3} = \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x} = \frac{x \cdot x \cdot x}{x \cdot x \cdot x} \cdot x^2 = 1 \cdot x^2 = x^{5-3}$$

three factors of $x$ in the numerator and denominator

5-3=2 factors of $x$

The **Zero Exponent Rule** defines the value of an expression with a power of 0.

**Zero Exponent Rule**

$$x^0 = 1$$

**Why it works**

Using the expanded form

$$\frac{x^3}{x^3} = \frac{x \cdot x \cdot x}{x \cdot x \cdot x} = 1$$

Using the Quotient of Powers Rule

$$\frac{x^3}{x^3} = x^{3-3} = x^0$$

Since $\frac{x^3}{x^3} = 1$ and $\frac{x^3}{x^3} = x^0$ then $x^0 = 1$

The **Negative Exponent Rule** is used to write expressions with negative exponents as expressions with positive exponents.

**Negative Exponent Rule**

$$x^{-a} = \frac{1}{x^a}$$

**Why it works**

Using the expanded form

$$\frac{x^3}{x^5} = \frac{x \cdot x \cdot x}{x \cdot x \cdot x \cdot x \cdot x} = \frac{x \cdot x \cdot x}{x \cdot x \cdot x} \cdot \frac{1}{x^2} = 1 \cdot \frac{1}{x^2} = \frac{1}{x^2}$$

Using the Quotient of Powers Rule

$$\frac{x^3}{x^5} = x^{3-5} = x^{-2}$$

Since $\frac{x^3}{x^5} = \frac{1}{x^2}$ and $\frac{x^3}{x^5} = x^{-2}$ then $x^{-2} = \frac{1}{x^2}$

For some expressions, several properties are needed. For example, the expression $\frac{x^4y^6}{x^3y^8}$:

| | |
|---|---|
| $\frac{x^4y^6}{x^3y^8}$ | The expressions are being divided, so subtract the exponents. |
| $= x^{4-2}y^{6-8}$ <br> $= xy^{-2}$ | The $y$ has a negative exponent, so rewrite it as the reciprocal with a positive exponent. |
| $= x \cdot \frac{1}{y^2}$ <br> $= \frac{x}{y^2}$ | Write the product as a single expression. |

**Quotient of Powers Rule**: When dividing two exponential expressions that have the same base, subtract the exponents: $\frac{x^a}{x^b} = x^{a-b}$

**Zero Exponent Rule**: $x^0 = 1$

**Negative Exponent Rule**: $x^{-a} = \frac{1}{x^a}$

## Cool-down: Powers, Part Two (*5 minutes*)

| |
|---|
| **Addressing:** NC.M1.N-RN.2 |
| **Cool-down Guidance:** More Chances<br>Students will continue to practice rewriting exponential expressions throughout the unit in practice problems. |

### Cool-down

Clare used the properties of exponents to rewrite the expression $\frac{8x^5y^{-2}}{2x^3}$. Examine her work and decide if you agree or not.

- If you agree, explain which exponent rule or calculation was applied at each step.
- If you disagree, explain the error and provide the correct equivalent expression.

$$\begin{aligned}\frac{8x^5y^{-2}}{2x^3} &= \frac{8x^5}{2x^3y^2}\\ &= \frac{8x^2}{2y^2}\\ &= \frac{4x^2}{y^2}\end{aligned}$$

**Student Reflection:**

Today's lesson included multiple opportunities for discussion. What made you more comfortable to share? If you were not comfortable, what would have supported you?

### DO THE MATH

### INDIVIDUAL STUDENT DATA

### SUMMARY DATA

## NEXT STEPS

## TEACHER REFLECTION

**What went well in the lesson? What would you do differently next time? What happened today that will influence the planning of future lessons?**

**In the previous lesson, you reflected on what you saw that you could leverage today for learning the Quotient of Powers Rule. Was your prediction correct? What surprised you about the learning you witnessed today?**

## Practice Problems

1. Rewrite each expression using the Zero Exponent Rule.
   a. $r^0$
   b. $5m^0$
   c. $(3xy)^0$
   d. $\frac{4rs}{s^0}$

2. Which of the following is equivalent to $\frac{12x^{10}y^{15}}{4x^5y^3}$?
   a. $3x^2y^5$
   b. $3x^5y^{12}$
   c. $8x^2y^5$
   d. $8x^5y^{12}$

3. Priya says, "I can figure out the value of $3^{-1}$ by looking at other powers of 3. I know that $3^2 = 9$, $3^1 = 3$, and $3^0 = 1$."
   a. What pattern do you notice?
   b. If this pattern continues, what should be the value of $3^{-1}$?
   c. If this pattern continues, what should be the value of $3^{-2}$?

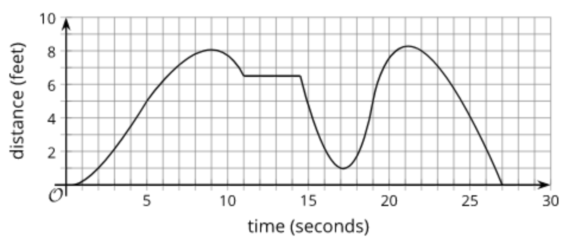
4. Kiran used the exponent rules to rewrite the expression $\frac{(5x^2y^4)^3}{x^{-4}y^6}$. His steps are shown below. For each step, identify which property was applied.

$$\frac{(5x^2y^4)^3}{x^{-4}y^6} = \frac{125x^6y^{12}}{x^{-4}y^6}$$
$$= 125x^{10}y^6$$

5. Rewrite each expression using the fewest number of exponents.
   a. $(4r^9)(3r^7)$
   b. $(5m^6n^3)^2$
   c. $(xy^2)(2x^3y^4)^3$

   (From Unit 6, Lesson 1)

6. Here is the graph of function $f$, which represents Andre's distance from his bicycle as he walked in a park.
   a. Estimate $f(5)$
   b. Estimate $f(17)$
   c. For what values of $t$ does $f(t) = 8$?
   d. For what values of $t$ does $f(t) = 6.5$?
   e. For what values of $t$ does $f(t) = 10$?




   (From Unit 5)

7. Two children set up a lemonade stand in their front yard. They charge \$1 for every cup. They sell a total of 15 cups of lemonade. The amount of money the children earned, $R$ dollars, is a function of the number of cups of lemonade they sold, $n$.

    a. Is 20 part of the domain of this function? Explain your reasoning.

    b. What does the range of this function represent?

    c. Describe the set of values in the range of $R$.

    d. Is the graph of this function discrete or continuous? Explain your reasoning.

    (From Unit 5)

8. Select **all** systems that are equivalent to this system of equations: $\begin{cases} 4x+5y=1 \\ x-y=\frac{3}{8} \end{cases}$

    a. $\begin{cases} 4x+5y=1 \\ 4x-4y=\frac{3}{2} \end{cases}$

    b. $\begin{cases} x+\frac{5}{4}y=\frac{1}{4} \\ x-y=\frac{3}{8} \end{cases}$

    c. $\begin{cases} 4x+5y=1 \\ 5x-5y=3 \end{cases}$

    d. $\begin{cases} 8x+10y=2 \\ 8x-8y=3 \end{cases}$

    e. $\begin{cases} x+y=\frac{1}{5} \\ x-y=\frac{3}{8} \end{cases}$

    (From Unit 3)

9. A catering company is setting up for a wedding. They expect 150 people to attend. They can provide small tables that seat 6 people and large tables that seat 10 people.

    a. Find a combination of small and large tables that seats exactly 150 people.

    b. Let $x$ represent the number of small tables and $y$ represent the number of large tables. Write an equation to represent the relationship between $x$ and $y$.

    c. Explain what the point $(20, 5)$ means in this situation.

    d. Is the point $(20, 5)$ a solution to the equation you wrote? Explain your reasoning.

    (From Unit 3)

# Lesson 3: Growing and Growing

## PREPARATION

| Lesson Goal | Learning Target |
|---|---|
| • Compare linear and exponential relationships by performing calculations and by interpreting graphs that show two growth patterns. | • I can compare growth patterns using calculations and graphs. |

### Lesson Narrative

The goal of this lesson is to encounter two different growth patterns—one pattern is linear and the other is exponential, though students don't need to use those words yet. Students think about and compare the patterns by performing calculations and using graphs. This lesson contains many opportunities for students to notice and make use of structure (MP7): for example, noticing that $1,000 + 200 + 200 + 200 + 200$ can be expressed as $1,000 + 200 \cdot 4$, which is useful when they must add on more and more $200$s. There is also an opportunity to use appropriate tools strategically (MP5): for example, if students choose to use a spreadsheet to perform many iterations of such calculations.

They see that the pattern, which grows by repeatedly doubling, starts off slowly but eventually overtakes the other pattern, which increases by repeatedly adding the same amount. In fact, the first pattern eventually leaves the second pattern far behind. Throughout the unit, students will study exponential patterns systematically (eventually viewing them as functions) before returning to this comparison with linear functions toward the end of the unit.

Some technology is required for this lesson, but there are opportunities for students to select appropriate technology in the lesson. It is recommended that scientific calculators and spreadsheet technology be made available.

**What teaching strategies will you be focusing on during this lesson?**

## Focus and Coherence

<table>
<tr><th>Building On</th><th>Addressing</th><th>Building Towards</th></tr>
<tr><td>NC.6.EE.1: Write and evaluate numerical expressions, with and without grouping symbols, involving whole-number exponents.<br><br>NC.6.EE.9: Represent and analyze quantitative relationships by:<br>● Using variables to represent two quantities in a real-world or mathematical context that change in relationship to one another.<br>● Analyze the relationship between quantities in different representations (context, equations, tables, and graphs)<br><br>NC.7.EE.3: Solve multi-step real-world and mathematical problems posed with rational numbers in algebraic expressions.<br>● Apply properties of operations to calculate with positive and negative numbers in any form.<br>● Convert between different forms of a number and equivalent forms of the expression as appropriate.<br><br>NC.8.F.5: Qualitatively analyze the functional relationship between two quantities.<br>● Analyze a graph determining where the function is increasing or decreasing; linear or non-linear.<br>● Sketch a graph that exhibits the qualitative features of a real-world function.</td><td>NC.M1.F-IF.4: Interpret key features of graphs, tables, and verbal descriptions in context to describe functions that arise in applications relating two quantities, including: intercepts; intervals where the function is increasing, decreasing, positive, or negative; and maximums and minimums.<br><br>NC.M1.F-LE.3: Compare the end behavior of linear, exponential, and quadratic functions using graphs and tables to show that a quantity increasing exponentially eventually exceeds a quantity increasing linearly or quadratically.</td><td>NC.M1.F-BF.1a: Build linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two ordered pairs (include reading these from a table).<br><br>NC.M1.A-CED.2: Create and graph equations in two variables to represent linear, exponential, and quadratic relationships between quantities.</td></tr>
</table>

## Agenda, Materials, and Preparation

- **Bridge** (*Optional, 5 minutes*)
- **Warm-up** (5 *minutes*)
- **Activity 1** (*15 minutes*)
- **Activity 2** (*10 minutes*)
- **Lesson Debrief** (*5 minutes*)
- **Cool-down** (*5 minutes*)
  - M1.U6.L3 Cool-down (print 1 copy per student)

## LESSON

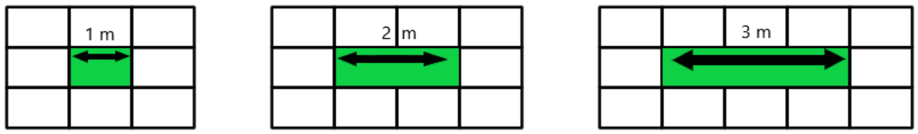
### Bridge (*Optional, 5 minutes*)

| **Building On:** NC.6.EE.9 | **Building Towards**: NC.M1.F-IF.4 |
|---|---|

The purpose of this bridge is for students to find the pattern of growth in a situation described verbally and in pictures. Students can organize the given information in a table and discuss how the table can help them see the pattern, and then reason to extend the pattern.

#### Student Task Statement

A homeowner wants to build a garden with concrete tiles around the outside. He has room for the garden to vary in length but not width. He's not sure what size he wants the garden to be. Here are sketches of gardens that are 1, 2, and 3 meters long. The homeowner needs to know how many concrete tiles will be needed for different possible garden lengths.




1. Create a table to show how many tiles will be needed if the garden is 1, 2, 3, 4, or 5 meters long.
2. Describe the way the pattern is growing.

**DO THE MATH**

**PLANNING NOTES**

## Warm-up: Two Expressions (*5 minutes*)

| | |
|---|---|
| **Instructional Routine:** Notice and Wonder | |
| **Addressing:** NC.6.EE.1 | **Building Towards:** NC.M1.F-BF.1.a |

This warm-up gives students an opportunity to notice and wonder about two different algebraic expressions that are present throughout this lesson.

### Step 1

- Display the two different algebraic expressions.
- Ask students to think of things they *Notice and Wonder* and write them in their workbook.
- Provide students 1 minute of quiet think time and then 1 minute to discuss the things they notice or wonder with their partner.

**Student Task Statement**

Here are two different algebraic expressions. What do you notice or wonder?

a. $70+5+5+5+5+5$ b. $70\cdot5\cdot5\cdot5\cdot5\cdot5$

### Step 2

- Facilitate a whole-class discussion by asking students to share the things they noticed and wondered. The goal of the discussion is to help students recognize that the two expressions look similar but are, in fact, very different in value.
- As students are sharing, listen for students verbalizing other ways to write the expressions. If this does not surface, ask students, “Can we write equivalent expressions for the given expressions?” Record and display their responses for all to see. (Sample responses: $70 + 5(5)$ and $70 \cdot 5^5$)
- After all responses have been recorded without commentary or editing, ask students, “Is there anything on this list that you are wondering about now?” Encourage students to respectfully disagree, ask for clarification, or point out contradicting information.

| DO THE MATH | PLANNING NOTES |
| --- | --- |
| | |

## Activity 1: A Genie in a Bottle (*15 minutes*)

| **Instructional Routines:** Poll the Class; Three Reads (MLR6) - Responsive Strategy | |
| --- | --- |
| **Building On:** NC.6.EE.1; NC.7.EE.3 | **Building Towards:** NC.M1.A-CED.2; NC.M1.F-BF.1a; NC.M1.F-LE.3 |

The goal of this activity is to allow students to explore two growth patterns and to notice that a doubling pattern eventually grows very, very large even with a small starting value. Students may perform a repeated calculation, or they may generalize the process by writing expressions or equations.

Making spreadsheet technology available gives students an opportunity to choose appropriate tools strategically (MP5).

### Step 1

- Present the situation in a lively manner without providing students with the task statement. Record important information for all students to see as necessary.

> You are walking along a beach and your toe hits something hard. You reach down, grab onto a handle, and pull out a lamp! It is sandy. You start to brush it off with your towel. Poof! A genie appears.
>
> He tells you, "Thank you for freeing me from that bottle! I was getting claustrophobic. You can choose one of these purses as a reward."
>
> - Purse A, which contains $1,000 today. If you leave it alone, it will contain $1,200 tomorrow (by magic). The next day, it will have $1,400. This pattern of $200 additional dollars per day will continue.
> - Purse B, which contains one penny today. Leave that penny in there, because tomorrow it will (magically) turn into two pennies. The next day, there will be four pennies. The amount in the purse will continue to double each day.

- *Poll the Class* about which option, purse A or purse B, they think is better. Display the results of the poll (the number of students who think purse A is better and purse B is better) for all to see.

### Step 2

- Ask students to arrange themselves in pairs, or use visibly random grouping.
- Provide pairs 5–7 minutes to work together to complete the three questions.

**RESPONSIVE STRATEGIES**

Represent the same information through different modalities by using diagrams. Encourage students to sketch diagrams that show how the amount of money grows over the first few days. Students may benefit from this visual as they transition to the use of a table or other representation to track growth.

Supports accessibility for: Conceptual processing; Visual-spatial processing

**RESPONSIVE STRATEGY**

Use this Three Reads routine to support reading comprehension of this word problem. Use the first read to orient students to the situation and address any unfamiliar words. Ask students to describe what the situation is about without using numbers (as a reward, students make a choice between purse A and purse B) and to identify any words they are unsure about. Provide definitions, examples, and visuals as needed to clarify meaning. Use the second read to identify quantities and relationships. Ask students what can be counted or measured without focusing on the values. Listen for, and amplify, the important quantities that vary in relation to each other in this situation: the amount of money, in dollars, and the amount of time, in days. After the third read, invite students to brainstorm possible strategies to answer the questions. This helps students connect the language in the word problem and the reasoning needed to solve the problem.

Three Reads (MLR6)

**Monitoring Tip:** Look for students who model the two situations in the following ways:

- performing recursive calculations ($1000 + 200 + 200 + ... + 200$ and $0.01 \cdot 2 \cdot 2 ... \cdot 2$)
- creating a table of values for purse A and purse B by day
- writing algebraic expressions ($1000 + 200(7)$ and $0.01(2)^7$)

***Advancing Student Thinking***: Some students may confuse the units for the two purses. Remind them that purse B contains pennies (and thus cents rather than dollars). Due to prior experience with genie stories, students may be suspicious of the offers. Ask them to show their mathematical reasoning, rather than basing their choice of purses on their suspicions or other considerations such as how to carry a purse containing over 2 million pennies.

### Student Task Statement

You are walking along a beach and your toe hits something hard. You reach down, grab onto a handle, and pull out a lamp! It is sandy. You start to brush it off with your towel. Poof! A genie appears.

He tells you, "Thank you for freeing me from that bottle! I was getting claustrophobic. You can choose one of these purses as a reward."

- Purse A, which contains $1,000 today. If you leave it alone, it will contain $1,200 tomorrow (by magic). The next day, it will have $1,400. This pattern of $200 additional dollars per day will continue.
- Purse B, which contains one penny today. Leave that penny in there, because tomorrow it will (magically) turn into two pennies. The next day, there will be four pennies. The amount in the purse will continue to double each day.

1. How much money will be in each purse after a week? After two weeks?
2. The genie later added that he will let the money in each purse grow for three weeks. How much money will be in each purse then?
3. Which purse contains more money after 30 days?

### Step 3

- Start the discussion by selecting students who performed recursive calculations to share the amounts of money in the two purses after one week.
- Next, ask students who modeled with a table to share. Ask students:
  - "What is similar in each of the ways students found the amount of money?" (Both repeatedly added 200 for purse A and repeatedly multiplied by 2 for purse B. For example, the amount in purse A will be $1,000 + 200$ for Day 1, then $1,000 + 200 + 200$ for Day 2, $1,000 + 200 + 200 + 200$ for Day 3, and so on.)
  - "How are the strategies presented different?" (The table shows the amount of money for each day, the recursive calculation jumps to the final amount for a specific day.)
- Finally ask students who constructed an expression for the purses to share their expressions for the amount of money in the purses after 30 days. If this strategy is not present in student work, extend the repeated reasoning from the recursive calculations and table to write expressions like $1,000 + 200 \cdot 3$ and $1,000 + 200x$ where $x$ represents the number of days since the genie appeared. Then ask the following questions:
  - "How do we know these expressions are equivalent to the other strategies used?" (Verify amounts for different days.)
  - "What is the benefit of using exponential notation in the expression for purse B?" (Exponential notation allows us to calculate the amount for any day without having to know the amount for all of the days before (the table) or multiply a larger number of times (the recursive sequence).)

| DO THE MATH | PLANNING NOTES |
| --- | --- |
| | |

## Activity 2: Graphing the Genie's Offer (*10 minutes*)

| **Instructional Routine:** Co-Craft Questions (MLR5) | |
| --- | --- |
| **Building On:** NC.8.F.5 | **Addressing:** NC.M1.F-IF.4; NC.M1.F-LE.3 |

This activity allows students to compare and contrast the two patterns visually, and to make sense of the graphs and use them to answer questions in the context of the genie's offer. They recall that a quantity that grows by adding the same amount at each step forms a line when plotted, and they see an example where a quantity that grows by the same factor at each step forms a curve when plotted.

### Step 1

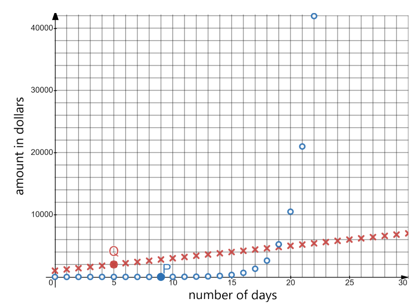
- Begin with the *Co-Craft Questions* routine by displaying graphs of the genie's two different offers and asking students, "What mathematical questions could be asked about the graphs of the genie's offers?"
- Give students 1–2 minutes individual time to jot down their own mathematical questions.

- Invite students to share their questions with the class. Listen for and amplify any questions that seek to determine which purse is the better choice or questions that reference critical points on the graphs.
- Connect relevant questions to the task statement and tell students they are going to further analyze the two offers graphically. Provide students 5 minutes to continue working with their partner from the previous activity.

***Advancing Student Thinking*:** Review the meaning of “vertical intercept” as needed. Due to the exponential nature of the problem, the scale for the vertical axis makes it difficult to accurately estimate vertical coordinates of points from the graph. Remind students that they know how the value of each purse is calculated, so they may use calculations similar to those in the previous activity to compute actual dollar values. If color versions of the materials are not available, it may be difficult to determine which points correspond with which purse. Encourage students to imagine connecting the dots (perhaps using a ruler for purse A) to distinguish the two sets of points.

### Student Task Statement

Here are graphs showing how the amount of money in the purses changes. Remember purse A starts with $1,000 and grows by $200 each day. Purse B starts with $0.01 and doubles each day.




1. Which graph shows the amount of money in purse A? Which graph shows the amount of money in purse B? Explain how you know.
2. Points $P(9, 5.12)$ and $Q(5, 2000)$ are labeled on the graph. Explain what they mean in terms of the genie’s offer.
3. What are the coordinates of the vertical intercept for each graph? Explain how you know.
4. When does purse B become a better choice than purse A? Explain your reasoning.
5. Knowing what you know now, which purse would you choose? Explain your reasoning.

### Are You Ready For More?

“Okay, okay,” the genie smiles, disappointed; “I will give you an even *more* enticing deal.” He explains that purse B stays the same, but purse A now increases by $250,000 every day. Which purse should you choose?

### Step 2

**RESPONSIVE STRATEGY**

Use color coding and annotations to highlight important connections between the representations for each situation (purse A or purse B).

Supports accessibility for: Conceptual processing; Visual-spatial processing

- Invite students to share their responses with the class. Discuss how they interpreted the points on the graphs and how they determined when the value of purse B surpassed the value of purse A.
- Ask students:
  - “You have used calculations and graphs to compare the genie's offers. Did using graphs help? Did they add new insights to the offers?” (The graphs give a clear visual representation of how the amount of money in each purse grows. It is easier to tell when purse B becomes a better option than purse A, and how different the two values are at different points in time. They de-emphasize the actual numbers and so give a very clean comparison of the two situations.)
- Consider asking students to explain to a partner their response and reasoning to the last question: “Knowing what you know now, which purse would you choose? Why?” If time permits, return to the poll the class results from Activity 1.

### DO THE MATH

### PLANNING NOTES

## Lesson Debrief (*5 minutes*)

The purpose of this lesson is for students to think about the differences between linear and exponential growth, and to realize that quantities that increase through repeated multiplication by the same factor will eventually exceed quantities that repeatedly increase by the same amount.

Choose whether students should first have an opportunity to reflect in their workbooks or talk through these questions with a partner.

Display this situation:

Another genie offers two purses:

Purse C: has $4,000 today and increases by $500 every day.

Purse D: has $1 today and triples every day after that ($3 on Day 1, $9 on Day 2…).

- "What are some ways we can calculate the value for purse D for any given day?" (We could do a lot of multiplication or we could make a table, but the quickest way would be to use an exponential expression such as $1 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 1 \cdot 3^4$.)
- "How would you expect the graphs for the two purses to be different?" (The graph for purse C would be a line. The graph for purse D would curve upward.)
- "Which option would you expect to pay more 'in the long run'?" (Purse D, because even though it starts with a smaller amount of money, the amount of money eventually starts growing faster than $500 per day.)

Students may disagree about the answer to the third question, which is okay for now.

### PLANNING NOTES

### Student Lesson Summary and Glossary

When we repeatedly double a positive number, it eventually becomes *very* large. Let's start with $0.001$. The table shows what happens when we begin to double:

| | | | | |
|---|---|---|---|---|
| $0.001$ | $0.002$ | $0.004$ | $0.008$ | $0.016$ |

If we want to continue this process, it is convenient to use an exponent. For example, the last entry in the table, $0.016$, comes from $0.001$ being doubled $4$ times, or $(0.001) \cdot 2 \cdot 2 \cdot 2 \cdot 2$, which can be expressed as $(0.001) \cdot 2^4$.

Even though we started with a very small number, $0.001$, we don't have to double it that many times to reach a very large number. For example, if we double it $30$ times, represented by $(0.001) \cdot 2^{30}$, the result is greater than $1,000,000$.

Throughout this unit, we will look at many situations where quantities grow or decrease by applying the same factor repeatedly.

## Cool-down: Rival Genie (*5 minutes*)

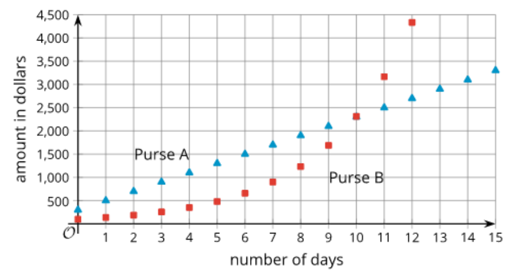
**Addressing:** NC.M1.F-IF.4

**Cool-down Guidance:** More Chances
Students will have more opportunities to understand the mathematical ideas in this cool-down, so there is no need to slow down or add additional work to the next lessons. Instead, use the results of this cool-down to provide guidance for what to look for and emphasize over the next several lessons to support students in advancing their current understanding.

### Cool-down

While on a beach, your friend discovers a different genie. This genie also offers two purses to choose from, and he gives you the following graph to show how the money in each purse will grow. The set of triangular marks that lie on a line represent values in purse A, and the set of square marks that make a curve represent those in purse B.

The genie is still deciding how many days he will let the money in the purses grow. Help your friend plan a strategy for picking the purse with the most money depending on the genie's answer.




**Student Reflection:**

If a genie could grant any change you wanted or needed in your math class, what would you ask the genie to change?

### DO THE MATH

### INDIVIDUAL STUDENT DATA

### SUMMARY DATA

## NEXT STEPS

## TEACHER REFLECTION

**What went well in the lesson? What would you do differently next time? What happened today that will influence the planning of future lessons?**

**What evidence have students given that they understand exponential notation? What language do they use or associate with exponential notation? Is there anything more you feel you should do to help improve understanding?**

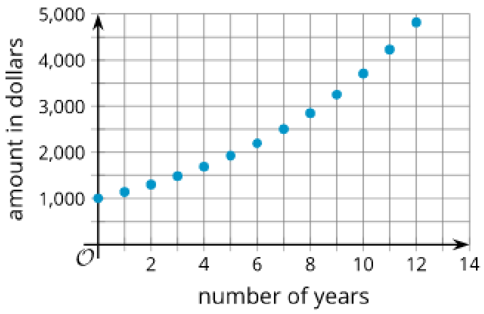
## Practice Problems

1. Which expression equals $2^7$?

   a. $2+2+2+2+2+2+2$
   b. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
   c. $2 \cdot 7$
   d. $2+7$

2. Evaluate the expression $3 \cdot 5^x$ when $x$ is $2$.

3. The graph shows the yearly balance, in dollars, in an investment account.




   a. What is the initial balance in the account?

   b. Is the account growing by the same number of dollars each year? Explain how you know.

   c. A second investment account starts with $2,000 and grows by $150 each year. Sketch the values of this account on the graph.

   d. How does the growth in the two account balances compare?

4. Jada rewrites $5 \cdot 3^x$ as $15x$. Do you agree with Jada that these are equivalent expressions? Explain your reasoning.

5. Investment account 1 starts with a balance of $200 and doubles every year. Investment account 2 starts with $1,000 and increases by $100 each year.

   a. How long does it take for each account to double?

   b. How long does it take for each account to double again?

   c. How does the growth in these two accounts compare? Explain your reasoning.

6. Write the following expression with only negative exponents: $(x^5y^{-2}x^8)^3$

   (From Unit 6, Lesson 2)

7. Lin says that a snack machine is like a function because it outputs an item for each code input. Explain why Lin is correct.

   (From Unit 5)

8. At a gas station, a gallon of gasoline costs $3.50. The relationship between the dollar cost of gasoline and the number of gallons purchased can be described with a function.

   a. Identify the input variable and the output variable in this function.

   b. Describe the function with a sentence of the form "_____is a function of _____."

   c. Identify an input-output pair of the function and explain its meaning in this situation.

   (From Unit 5)

9. Noah and Lin are solving this system: $\begin{cases} 8x + 15y = 58 \\ 12x - 9y = 150 \end{cases}$

   Noah multiplies the first equation by 12 and the second equation by 8, which gives: $\begin{cases} 96x + 180y = 696 \\ 96x - 72y = 1,200 \end{cases}$

   Lin says, "I know you can eliminate $x$ by doing that and then subtracting the second equation from the first, but I can use smaller numbers. Instead of what you did, try multiplying the first equation by 6 and the second equation by 4."

   a. Do you agree with Lin that her approach also works? Explain your reasoning.

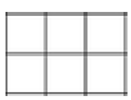
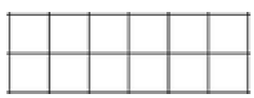
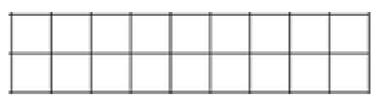
   b. What are the smallest whole-number factors by which you can multiply the equations in order to eliminate $x$?

   (From Unit 3)

10. Solve this system of linear equations without graphing: $\begin{cases} 7x + 11y = -2 \\ 7x + 3y = 30 \end{cases}$

    (From Unit 3)

11. Each small square below represents 1 square unit.

| **Figure 1** | **Figure 2** | **Figure 3** | **Figure 4** | **Figure 5** |
|---|---|---|---|---|
|  |  |  | | |
| | | | | |

   a. Indicate the number of squares in figures 1, 2, 3, 4, and 5.

   b. Describe how the pattern is growing.

   c. Write an equation representing the number of boxes, $y$, in figure $x$.

   (Addressing NC.6.EE.9)

# Lesson 4: Patterns of Growth

## PREPARATION

| Lesson Goals | Learning Targets |
|---|---|
| • Describe (using words and expressions) patterns in tables that represent linear and exponential relationships.<br>• Given descriptions of linear and exponential relationships, create tables of values and write expressions. | • I can use words and expressions to describe patterns in tables of values.<br>• When I have descriptions of different types of growth relationships between two quantities, I can write expressions and create tables of values to represent them. |

### Lesson Narrative

In this lesson, students continue to examine two types of patterns by looking at tables, noting that one type increases by a constant rate while the other increases by multiplying by the same factor. They then match different representations with two contexts. This is an opportunity for students to notice structure in expressions (MP7). Note we are still not expecting students to use the words “linear” or “exponential” to describe these patterns, though they may naturally use “linear” to describe linear growth, since they have encountered these types of functions before.

**What math language will you want to support your students with in this lesson? How will you do that?**

### Focus and Coherence

| Building On | Addressing | Building Towards |
|---|---|---|
| **NC.7.RP.3:** Use scale factors and unit rates in proportional relationships to solve ratio and percent problems. | **NC.M1.F-IF.7:** Analyze linear, exponential, and quadratic functions by generating different representations, by hand in simple cases and using technology for more complicated cases, to show key features, including: domain and range; rate of change; intercepts; intervals where the function is increasing, decreasing, positive, or negative; maximums and minimums; and end behavior. | **NC.M1.F-BF.1a:** Build linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two ordered pairs (include reading these from a table).<br><br>**NC.M1.A-CED.2:** Create and graph equations in two variables to represent linear, exponential, and quadratic relationships between quantities.<br><br>**NC.M1.F-LE.1:** Identify situations that can be modeled with linear and exponential functions, and justify the most appropriate model for a situation based on the rate of change over equal intervals. |

## Agenda, Materials, and Preparation

- **Bridge** (*Optional, 5 minutes*)
- **Warm-up** (*5 minutes*)
- **Activity 1** (*15 minutes*)
- **Activity 2** (*10 minutes*)
- **Lesson Debrief** (*5 minutes*)
- **Cool-down** (*5 minutes*)
    - M1.U6.L4 Cool-down (print 1 copy per student)

# LESSON

## Bridge (*Optional, 5 minutes*)

**Building On:** NC.7.RP.3

This bridge reviews finding a common ratio in a proportional relationship. This sets students up for the work of this lesson as they calculate the growth factor from one year to the next.

### Student Task Statement

Tyler and Mai purchased gas from the same gas station. Tyler purchased 4 gallons for a total cost of $14. Mai purchased 12 gallons for a total cost of $42. What is the price per gallon?

| DO THE MATH | PLANNING NOTES |
|---|---|
| | |

## Warm-up: Tables of Values (*5 minutes*)

**Instructional Routine:** Which One Doesn't Belong?

**Building Towards:** NC.M1.F-BF.1a; NC.M1.F-LE.1

This warm-up prompts students to compare four tables using the *Which One Doesn't Belong?* routine. It gives students a reason to use language precisely (MP6) and gives the opportunity to hear how they use terminology and talk about characteristics of the items in comparison to one another. Encourage students to find reasons based on mathematical properties.

This warm-up prepares students to analyze patterns of linear and exponential functions in a table.

### Step 1

- Ask students to arrange themselves in small groups or use visibly random grouping.

- Display the tables for all to see.
- Give students 1 minute of think time and then time to share their thinking with their small group. In their small groups, ask each student to share their reasoning as to why a particular table does not belong, and together find at least one reason each table doesn't belong.

### Student Task Statement

Which one doesn't belong? Explain your reasoning.

Table A

| | | | | | |
|---|---|---|---|---|---|
| $x$ | 1 | 2 | 3 | 4 | 8 |
| $y$ | 8 | 16 | 24 | 32 | 64 |

Table B

| | | | | | |
|---|---|---|---|---|---|
| $x$ | 0 | 2 | 4 | 6 | 8 |
| $y$ | 0 | 16 | 32 | 48 | 64 |

Table C

| | | | | | |
|---|---|---|---|---|---|
| $x$ | 0 | 1 | 2 | 3 | 4 |
| $y$ | 1 | 4 | 16 | 64 | 256 |

Table D

| | | | | | |
|---|---|---|---|---|---|
| $x$ | 0 | 1 | 2 | 3 | 4 |
| $y$ | 4 | 8 | 12 | 16 | 20 |

### Step 2

- Ask each group to share one reason why a particular table does not belong.
- Record and display the responses for all to see. After each response, ask the class if they agree or disagree. Since there is no single correct answer to the question of which one does not belong, attend to students' explanations and ensure the reasons given make sense to the class.
- During the discussion, students may say that table C doesn't belong because the pattern in the $y$-values is multiplying by 4. It may be useful to rephrase to help students connect that a factor is used as a multiplier. For example, "The $y$-values are multiplied by a factor of 4."

**DO THE MATH**

**PLANNING NOTES**

## Activity 1: Growing Stores (*15 minutes*)

| Instructional Routines: Notice and Wonder; Compare and Connect (MLR7) | |
|---|---|
| **Addressing:** NC.M1.F-IF.7 | **Building Towards:** NC.M1.A-CED.2; NC.M1.F-BF.1a; NC.M1.F-LE.1 |

In the previous lesson, students compared two patterns presented via a description and a graph. In this activity, they make the same kind of comparisons using a table. A table is particularly helpful for illustrating that:

- linear functions grow by equal differences over equal intervals, and
- exponential functions grow by equal factors over equal intervals.

At this stage, students simply observe these properties for a few particular functions and in the context of concrete problems. Toward the end of the unit, they will prove these properties are true for any linear or exponential function.

### Step 1

- Display the following table for all to see. It shows the number of coffee shops worldwide that a company owned in each of its first 10 years. Ask students what they *Notice and Wonder*. This routine should be quick and is included to help students make sense of the context. Invite students to share what they noticed and wondered about.
    - Students may notice that the number of stores nearly doubled between 1987 and 1988. The number continued to grow by more than 1.5 times per year (except from 1990 to 1991 and from 1991 to 1992) or that they added close to 1,000 stores in 10 years.
    - They may wonder:
        - How many stores do they have now? (22,519 in 2015, more than 33,000 in 2021)
        - Have the stores increased by roughly the same number since 1996?
        - How was the company able to add so many stores each year?

| Year | Number of stores |
|---|---|
| 1987 | 17 |
| 1988 | 33 |
| 1989 | 55 |
| 1990 | 84 |
| 1991 | 116 |
| 1992 | 165 |
| 1993 | 272 |
| 1994 | 425 |
| 1995 | 677 |
| 1996 | 1,015 |
| ... | |
| 2015 | |

### Step 2

- Tell students that they will now continue to work with their partner or small group to look at a couple of possible ways a company can expand its business.

***Advancing Student Thinking***: Students may struggle to write an expression using the variable $n$. Ask them to tell you any patterns they notice in the tables and then encourage them to use those patterns to write an expression involving $n$.

**RESPONSIVE STRATEGIES**

Review the terms "difference" and "factor" with students. In addition, help students get started by providing the calculations for "difference "and for "factor" (the first row of each table).

## Student Task Statement

A food company currently has five convenience stores. It is considering two plans for expanding its chain of stores.

1. Plan A: Open 20 new stores each year.
    a. Use technology to complete a table for the number of stores for the next 10 years, as shown here.
    b. What do you notice about the difference from year to year?
    c. If there are $n$ stores one year, how many stores will there be a year later?

| Year | Number of stores | Difference from previous year |
|---|---|---|
| 0 | 5 | |
| 1 | 25 | |
| 2 | | |
| 3 | | |
| 4 | | |
| 5 | | |
| 6 | | |

2. Plan B: Double the number of stores each year.
    a. Use technology to complete a table for the number of stores for the next 10 years under each plan, as shown here.
    b. What do you notice about the difference from year to year?
    c. What do you notice about the factor from year to year?
    d. If there are $n$ stores one year, how many stores will there be a year later?

| Year | Number of stores | Difference from previous year | Factor from previous year |
|---|---|---|---|
| 0 | 5 | | |
| 1 | | | |
| 2 | | | |
| 3 | | | |
| 4 | | | |
| 5 | | | |
| 6 | | | |

## Are You Ready For More?

Suppose the food company decides it would like to grow from the five stores it has now so that 5 years from now, it will have at least 600 stores but no more than 800 stores.

1. Come up with a plan for the company to achieve this where it adds the same number of stores each year.
2. Come up with a plan for the company to achieve this where the number of stores increases by the same factor each year. (Note that you might need to round the outcome to the nearest whole store.)

### Step 3

- Have students compare and contrast the two tables by asking:
    - “How are the tables alike? Or how was the process of completing the two tables the same?” (The tables both follow a simple pattern: each table row is calculated by applying the same operation to the previous row. The entries in both tables increase as we go down the table.)
    - “How are the tables different? Or how was the process of completing the two tables different?” (In plan A, 20 new stores are *added* each year; the repeated operation is addition. In plan B, the number of stores is *doubled* each year; the repeated operation is multiplication. Plan B grows much more quickly than plan A.)

Use the *Compare and Connect* routine to call students’ attention to the related mathematical operations within the context of plan A and plan B. For example, ask students, “Why does one expression include multiplication while the other does not?”

- Find opportunities to amplify the meanings of “difference from previous year” and “factor from previous year” while highlighting connections to the related expressions.

– Discuss how students went about finding the factor from the previous year. For plan B, students may notice in the table that the difference from the number of stores the previous year is the same as the number of stores the previous year. Prompt students to explain why in their own words, and revoice as needed. (Because of the doubling pattern, the number from the previous year is added to itself (or doubled).)

| DO THE MATH | PLANNING NOTES |
| --- | --- |
| | |

## Activity 2: Flow and Followers (*10 minutes*)

Students study two stories characterized by different growth patterns, and they match expressions and tables with the two situations. The numbers in the two scenarios are identical, so students need to focus on the mathematical relationships and the operations in the expressions to successfully complete the activity. If students are evaluating the expressions, encourage them to reason about the operations without calculating anything.

| **Instructional Routine:** Discussion Supports (MLR8) - Responsive Strategy | |
| --- | --- |
| **Addressing:** NC.M1.F-IF.7 | **Building Towards:** NC.M1.A-CED.2; NC.M1.F-BF.1a; NC.M1.F-LE.1 |

### Step 1

- Have students continue to work with their small groups or partners.
- Tell students they have two different situations (situations 1 and 2) along with multiple representations of each situation. Students should take turns matching a representation to the situation without performing any calculations and explaining the match to their partner.

**RESPONSIVE STRATEGY**

Use color coding and annotations to highlight important connections between the text of each situation and the corresponding tables and expressions. Begin by asking students what they noticed to help identify how the difference and factors appear in each representation.

Supports accessibility for: Visual-Spatial Processing; Conceptual processing

**Monitoring Tip:** As students are explaining their matches to their partner, listen for students justifying the matches by identifying those that multiply by the same factor and those that are adding the same difference and connecting them to the representation(s).

## Student Task Statement

Here are verbal descriptions of two situations, followed by tables and expressions that could help to answer one of the questions in the situations.

- Situation 1: A person has 80 followers on social media. The number of followers triples each year. How many followers will she have after 4 years?
- Situation 2: A tank contains 80 gallons of water and is getting filled at a rate of 3 gallons per minute. How many gallons of water will be in the tank after 4 minutes?

Match each representation (a table or an expression) with situation 1 or situation 2. Be prepared to explain how the table or expression answers the question.

a. $8 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

b.

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $y$ | 80 | 240 | 720 | 2,160 | 6,480 |

c. $80 + 3 + 3 + 3 + 3$

d. $80 + 4 \cdot 3$

e.

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $y$ | 80 | 83 | 86 | 89 | 92 |

f. $80 \cdot 81$

g. $80 + 12$

h. $80 \cdot 3^4$

### Step 2

Focus the discussion on how students went about matching the cards and the situations. Record students' reasoning for all to see. In particular, highlight observations about common differences and common factors. Ask questions such as:

- "Besides evaluating the expression, how could you tell (a certain expression) matched one of the situations?"
- "How could you tell (a certain table) matched one of the situations?"
- "How is the number of followers growing every 2 years?"
- "How is the amount of water in the tank increasing every 2 minutes?"

**RESPONSIVE STRATEGY**

Use this routine support students to restate what they heard using their own everyday language. For each observation that is shared, ask students to restate what they heard using precise mathematical language. Consider providing students time to restate what they hear to a partner before selecting one or two students to share with the class. Ask the original speaker if their peer was accurately able to restate their thinking. Call students' attention to any words or phrases that helped to clarify the original statement. This provides more students with an opportunity to produce language as they interpret the reasoning of others.

Discussion Supports (MLR8)

## DO THE MATH

## PLANNING NOTES

## Lesson Debrief (*5 minutes*)

The purpose of this lesson is to compare linear and exponential growth patterns, beginning with tables and descriptions.

Choose what questions to focus the discussion on, whether students should first have an opportunity to reflect in their workbooks or talk through these with a partner, and what questions will be prioritized in the full class discussion.

Facilitate a class discussion using tables and situations from a classroom activity, or new tables and situations such as these. Ask the following questions:

This table shows the height, in centimeters, of water in a swimming pool as it is being filled.

| **Time in minutes** | **Height in centimeters** |
|---|---|
| 0 | 50 |
| 1 | 53 |
| 2 | 56 |
| 3 | 59 |
| 4 | 62 |

This table shows the total number of possible outcomes (sequences of heads or tails) when flipping a certain number of coins.

| **Number of coins** | **Number of outcomes** |
|---|---|
| 1 | 2 |
| 2 | 4 |
| 3 | 8 |
| 4 | 16 |
| 5 | 32 |

- "How are the values in each table growing?" (The table of the height of water in a swimming pool is growing by the same difference of 3 for every minute. The number of outcomes of flipping a coin is multiplying by the same factor of 2 for each new coin.)
- "Which table shows adding the same difference? Which one shows multiplying by the same factor?" (The height of water in the swimming pool by minute shows adding the same difference. The number of outcomes for coin sequences shows multiplying by the same factor.)
- "What expression can we write to find the height of the water after 10 minutes?" ($50 + 3 \cdot 10$)
- "What expression can we write to find the number of possible outcomes for 7 coins?" $(1 \cdot 2^7)$

## PLANNING NOTES

## Student Lesson Summary and Glossary

Here are two tables representing two different situations.

- A student runs errands for a neighbor every week. The table shows the pay he may receive, in dollars, in any given week.

| Number of errands | Pay in dollars | Difference from previous pay |
|---|---|---|
| 0 | 10 | |
| 1 | 15 | $15 - 10 = 5$ |
| 2 | 20 | $20 - 15 = 5$ |
| 3 | 25 | $25 - 20 = 5$ |
| 4 | 30 | $30 - 25 = 5$ |

- A student at a high school heard a rumor that a celebrity will be speaking at graduation. The table shows how the rumor is spreading over time, in days.

| Day | People who have heard the rumor | Factor from previous number of people |
|---|---|---|
| 0 | 1 | |
| 1 | 5 | $5 \div 1 = 5$ |
| 2 | 25 | $25 \div 5 = 5$ |
| 3 | 125 | $125 \div 25 = 5$ |
| 4 | 625 | $625 \div 125 = 5$ |

Once we recognize how these patterns change, we can describe them mathematically. This allows us to understand their behavior, extend the patterns, and make predictions.

In upcoming lessons, we will continue to describe and represent these patterns and use them to solve problems.

## Cool-down: Meow Island and Purr Island (*5 minutes*)

**Addressing:** NC.M1.F-IF.7

**Cool-down Guidance:** More Chances
Students will have more opportunities to understand the mathematical ideas in this cool-down, so there is no need to slow down or add additional work to the next lessons. Instead, use the results of this cool-down to provide guidance for what to look for and emphasize over the next several lessons to support students in advancing their current understanding.

## Cool-down

The tables show the cat population on two islands over several years. Describe mathematically, as precisely as you can, how the cat population on each island is changing.

| **Year** | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| **Number of cats on Meow Island** | 2 | 6 | 18 | 54 | 162 |

| **Year** | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| **Number of cats on Purr Island** | 2 | 6 | 10 | 14 | 18 |

**Student Reflection:**

Which activity was most helpful in your understanding today? Why?

## DO THE MATH

## INDIVIDUAL STUDENT DATA

## SUMMARY DATA

## NEXT STEPS

## TEACHER REFLECTION

**What went well in the lesson? What would you do differently next time? What happened today that will influence the planning of future lessons?**

**What opportunities are you giving students to reflect on their understanding of the mathematical content?**

## Practice Problems

1. A population of ants was 10,000 at the start of April. Since then, it has tripled each month.

    a. Complete the table.

    b. What do you notice about the population differences from month to month?

    c. If there are $n$ ants one month, how many ants will there be a month later?

| Months since April | Number of ants |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| 4 | |

2. A swimming pool contains 500 gallons of water. A hose is turned on, and it fills the pool at a rate of 24 gallons per minute. Which expression represents the amount of water in the pool, in gallons, after 8 minutes?

    a. $500 \cdot 24 \cdot 8$
    b. $500 + 24 + 8$
    c. $500 + 24 \cdot 8$
    d. $500 \cdot 24^8$

3. The population of a city is 100,000. It doubles each decade for 5 decades. Select **all** expressions that represent the population of the city after 5 decades.

    a. $32,000$
    b. $320,000$
    c. $100,000 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
    d. $100,000 \cdot 5^2$
    e. $100,000 \cdot 2^5$

4. The table shows the height, in centimeters, of the water in a swimming pool at different times since the pool started to be filled.

    a. Does the height of the water increase by the same amount each minute? Explain how you know.

    b. Does the height of the water increase by the same factor each minute? Explain how you know.

| Minutes | Height (centimeters) |
|---|---|
| 0 | 150 |
| 1 | 150.5 |
| 2 | 151 |
| 3 | 151.5 |

5. Bank account C starts with \$10 and doubles each week. Bank account D starts with \$1,000 and grows by \$500 each week. When will account C contain more money than account D? Explain your reasoning.

    (From Unit 6, Lesson 3)

6. Rewrite each expression using the fewest number of exponents. Show or explain your reasoning.

   a. $(x^6y^5)^2(2x3y^2)$

   b. $\frac{12m^4n^2}{3m^6n}$

   c. $(5x^{-3})(5x^0)(5x)$

   (From Unit 6, Lessons 1 and 2)

7. Suppose $C$ is a rule that takes time as the input and gives your class on Monday as the output. For example, $C(10:15) =$ Biology.

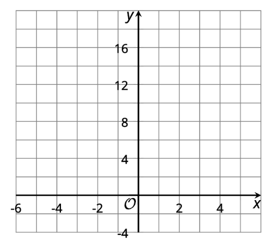
   a. Write three sample input-output pairs for $C$.

   b. Does each input to $C$ have exactly one output? Explain how you know.

   c. Explain why $C$ is a function.

   (From Unit 5)

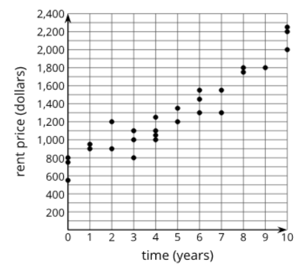
8. The rule that defines function $f$ is $f(x) = x^2 + 1$. Complete the table. Then, sketch a graph of function $f$.




| $x$ | $f(x)$ |
|---|---|
| -4 | 17 |
| -2 | |
| 0 | |
| 2 | |
| 4 | |

   (From Unit 5)

9. The scatter plot shows the rent prices for apartments in a large city over ten years.




   a. The regression equation is $y = 134.02x + 655.40$, where $y$ represents the rent price in dollars, and $x$ the time in years. Use it to estimate the rent price after 8 years. Show your reasoning.

   b. Use the best fit line to estimate the number of years it will take the rent price to equal $2,500. Show your reasoning.

   (From Unit 4)

10. One horse can run $\frac{3}{4}$ of a mile in $1.5$ minutes. Another runs $2\frac{1}{4}$ miles in $4.5$ minutes. Assuming the horses run at a constant speed, how long does it take a horse to run $1$ mile?

   (Addressing NC.7.RP.3)

# Lesson 5: Representing Exponential Growth

## PREPARATION

| Lesson Goals | Learning Targets |
|---|---|
| • Explain (in writing) how to see $a$ and $b$ on the graph of an equation of the form $y = a \cdot b^x$.<br>• Interpret $a$ and $b$ given equations of the form $y = a \cdot b^x$ and a context of exponential growth.<br>• Write an equation of the form of $y = a \cdot b^x$ to represent a quantity $a$ that changes by a growth factor $b$. | • I can explain the connections between an equation and a graph that represents exponential growth.<br>• I can write and interpret an equation that represents exponential growth. |

### Lesson Narrative

In this lesson, students study a situation characterized by **exponential growth** and learn the term **growth factor**. They represent this relationship using a table, an expression, and a graph. They also explain the meaning of the numbers $a$ and $b$ in an exponential expression $a \cdot b^x$, identifying their meaning in terms of a context ($a$ is the initial amount and $b$ is the multiplier. When $b > 1$, it is also known as a growth factor and, introduced in the next lesson, when $b < 1$, it is also known as a decay factor) and also in terms of a graph (where $a$ is the vertical intercept and $b$ determines how quickly the graph increases). Students interpret the different representations of growth in terms of a bacteria population (MP2).

In this and following lessons, students will often work with properties of exponents. There is an optional activity intended to remind students of the convention that $a^0 = 1$ for a non-zero number $a$.

While technology isn't required for this lesson, there are opportunities for students to choose to use appropriate technology to solve problems. It is recommended that technology be made available.

**What strategies or representations do you anticipate students might use in this lesson?**

## Focus and Coherence

| Building On | Addressing | Building Towards |
|---|---|---|
| **NC.6.EE.2:** Write, read, and evaluate algebraic expressions.<br>• Write expressions that record operations with numbers and with letters standing for numbers.<br>• Identify parts of an expression using mathematical terms and view one or more of those parts as a single entity.<br>• Evaluate expressions at specific values of their variables using expressions that arise from formulas used in real-world problems. | **NC.M1.A-CED.2:** Create and graph equations in two variables to represent linear, exponential, and quadratic relationships between quantities.<br><br>**NC.M1.A-SSE.1a:** Interpret expressions that represent a quantity in terms of its context. a. Identify and interpret parts of a linear, exponential, or quadratic expression, including terms, factors, coefficients, and exponents.<br><br>**NC.M1.F-LE.5:** Interpret the parameters $a$ and $b$ in a linear function $f(x) = ax + b$ or an exponential function $g(x) = ab^x$ in terms of a context.<br><br>**NC.M1.F-IF.7:** Analyze linear, exponential, and quadratic functions by generating different representations, by hand in simple cases and using technology for more complicated cases, to show key features, including: domain and range; rate of change; intercepts; intervals where the function is increasing, decreasing, positive, or negative; maximums and minimums; and end behavior. | **NC.M1.F-BF.1a:** Build linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two ordered pairs (include reading these from a table). |

## Agenda, Materials, and Preparation

- **Bridge** (*Optional, 5 minutes*)
- **Warm-up** (*5 minutes*)
- **Activity 1** (*10 minutes*)
- **Activity 2** (*15 minutes*)
  - Technology is suggested for this activity: Acquire devices that can access Desmos (recommended) or other graphing technology. It is ideal if each student has their own device.
- **Lesson Debrief** (*5 minutes*)
- **Cool-down** (*5 minutes*)
  - M1.U6.L5 Cool-down (print 1 copy per student)

## LESSON

### Bridge (*Optional, 5 minutes*)

**Building On:** NC.6.EE.2

In this bridge, students revisit using the order of operations to evaluate expressions with exponents. This task is aligned to question 2 in Check Your Readiness.

#### Student Task Statement

Evaluate each expression.[1]

1. $7 + 2^3$
2. $9 \cdot 3^1$
3. $20 - 2^4$
4. $2 \cdot 6^2$
5. $8 \cdot (\frac{1}{2})^2$
6. $\frac{1}{3} \cdot 3^3$
7. $(\frac{1}{5} \cdot 5)^5$

[1] Adapted from IM 6–8 Math https://curriculum.illustrativemathematics.org/MS/index.html, which was originally developed by Open Up Resources and authored by Illustrative Mathematics, and is copyright 2017–2019 by Open Up Resources. It is licensed under the Creative Commons Attribution 4.0 International License (CC BY 4.0). OUR's 6–8 Math Curriculum is available at https://openupresources.org/math-curriculum/. Adaptations and updates to IM 6–8 Math are copyright 2019 by Illustrative Mathematics, and are licensed under the Creative Commons Attribution 4.0 International License (CC BY 4.0).

**DO THE MATH**

**PLANNING NOTES**

## Warm-up: Splitting Bacteria (*5 minutes*)

**Building Towards:** NC.M1.A-CED.2; NC.M1.F-BF.1a

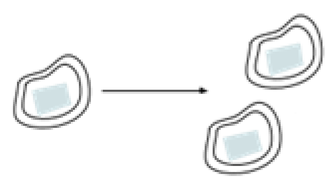
This warm-up gives students an opportunity to create a pictorial representation of exponential growth. The purpose of this warm-up is to help students make sense of the situation coming up in the next activity, where they are not explicitly asked to make a pictorial representation.

- Ask students to complete the task.
- Invite a few students to share their drawings and their observations about how the number of bacteria is growing.

### Student Task Statement

There are some bacteria in a dish. Every hour, each bacterium splits into two bacteria.

1. This diagram shows the number of bacteria in hour 0 and then in hour 1. Draw what happens in hours 2 and 3.
2. How many bacteria are there in hours 2 and 3?



**DO THE MATH**

**PLANNING NOTES**

## Activity 1: HeLa Cells (*10 minutes*)

| **Instructional Routine:** Collect and Display (MLR2) | |
|---|---|
| **Addressing:** NC.M1.A-CED.2; NC.M1.F-LE.5; NC.M1.F-IF.7; NC.M1.A-SSE.1a | **Building Towards:** NC.M1.F-BF.1a |

The context for this activity, cells taken from Henrietta Lacks for diagnosis and treatment that are still used today for research, will be elaborated on in Unit 6, Lessons 9 & 10, Station D. This activity prompts students to build expressions of the form $a \cdot b^x$ to encapsulate a type of pattern they have encountered several times so far, and to consider what $a$ and $b$ mean in the context of cell growth. They do so by writing numerical expressions that make explicit the key feature of exponential change, the repeated multiplication by the same factor, and then making a generalization of their repeated reasoning (MP8) using exponential notation. Since students are finally representing this pattern using an exponent, a quantity following this type of pattern is described as *changing exponentially*. The term “growth factor” is given to the multiplier or $b$ in an expression of the form $a \cdot b^x$, where $b > 1$.

### Step 1

- Ask students to read their task statement.
- If needed, provide an example (e.g., write the expression for the first day as $500 \cdot 2$ and evaluate in the third column).

### Step 2

- As students are working, use the *Collect and Display* routine. Listen for and collect vocabulary and phrases students use to describe patterns in the table for the growth of bacteria. Display words and phrases such as “multiplier,” “exponentially,” and “doubling,” for all students to see.
- Remind students to suggest additions and to borrow language from the display as needed. This will help students read and use mathematical language during their partner and whole-group discussions.

**RESPONSIVE STRATEGIES**

Chunk this task or the text into more manageable parts to aid students who benefit from support with organization and problem solving, or language. For example, students complete hours 0–3 and discuss the pattern they notice. Use annotations to show how the number of bacteria is changing from one hour to the next, then invite students to use what they notice to complete the table.

Supports accessibility for: Organization; Attention; Conceptual processing; Language

***Advancing Student Thinking***: For the first question, some students may write either $2 \cdot 500$ or $500 + 500$ for the number of bacteria after one hour. Both are mathematically correct, but $2 \cdot 500$ is more helpful for identifying a pattern, which will help generate an expression for the number of bacteria after $t$ hours. If they struggle to complete the table, refocus their attention on the second row of the table and ask them if there is a different expression they could use for the number of bacteria after one hour.

In the last row, students may write something like $500 \cdot 2 \cdot 2 \cdot ...2$ with a note about there being $t$ 2s. Encourage them to think how they might be able to write this expression more concisely.

### Student Task Statement

1. In a medical research lab, 500 HeLa cells double approximately every 24 hours (1 day).
    a. In the middle column, the expression to show how to find the number of HeLa cells after each day has been started for you. Complete each expression in the middle column.
    b. Evaluate your expression in the third column.
    c. Write an equation relating $n$, the number of cells, to $t$, the number of days.
    d. Use your equation to find $n$ when $t$ is 0. What does this value of $n$ mean in this situation?

| Day | Expression to determine the number of HeLa cells | Number of HeLa cells (evaluate your expression in the middle column) |
|---|---|---|
| 0 | 500 | |
| 1 | $500 \cdot$ | |
| 2 | $500 \cdot$ | |
| 3 | $500 \cdot$ | |
| 6 | $500 \cdot$ | |
| $t$ | $500 \cdot$ | |

2. In a different medical research lab, a population of single-cell parasites also reproduces. An equation which gives the number of parasites, $p$, after $t$ days is $p = 100 \cdot 3^t$. Explain what the numbers 100 and 3 mean in this situation.

### Step 3

- Invite students to share the expressions in their table and their generalized expression for the number of HeLa cells after $t$ hours. Make connections between, for example, $500 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$, the more concise expression $500 \cdot 2^5$, and the more general expression representing any number of hours $500 \cdot 2^t$. Highlight that $500$ is not only the initial number of cells but also the result of evaluating $500 \cdot 2^0$.
- Tell students that in patterns like these, where a quantity is repeatedly multiplied by the same factor, the quantity is often described as "changing exponentially." They can see why: an exponent is used to express the relationship. The term for the multiplier, 2 in the doubling relationship and 3 in the tripling relationship, is the "growth factor."

### Step 4

Questions for discussion: select one or two.

- "Is the growth of the HeLa cells characterized by common differences or common factors? How do you know?" (Common factors, since each time the number of days increases by 1, the number of cells is multiplied by the same factor.)
- "In each row in the table, what does the value of 500 mean? Why doesn't it change?" (It is the initial number of HeLa cells when they are first measured.)
- "What does $2^0$ mean in this situation?" ($2^0$ tells us no doubling has happened, so the original quantity of 500 HeLa cells is all we have.)
- "What do the 100 and 3 mean in the expression $100 \cdot 3^t$?" (100 is the initial population of the parasites when they are first measured, and the number 3 is the growth factor, the number by which the population is multiplied each hour.)
- "If the starting parasite population is 80 but the population quadruples every hour, how will the expression change?" (It will be $80 \cdot 4^t$.)

| DO THE MATH | PLANNING NOTES |
| --- | --- |
| | |

## Activity 2: Graphing the HeLa Cells (*15 minutes*)

| |
| --- |
| **Instructional Routines:** Graph It; Take Turns; Discussion Supports (MLR8) |
| **Addressing:** NC.M1.A-CED.2; NC.M1.F-IF.7 |

Having just seen an example of the meaning of $a$ and $b$ in an exponential expression $a \cdot b^x$ students now focus on interpreting these numbers using graphs in this *Graph It* activity. They graph the equations from the previous task, noticing that $a$ is the vertical intercept of the graph while the number $b$ determines how quickly the graph grows (since in these cases $b > 1$). A larger value of $b$ corresponds to a more rapid rate of growth for the HeLa cells. The axes for the graphs have been labeled here, but in future activities, students will have to think strategically about how to label the axes to most effectively plot the points.

Making graphing technology available gives students an opportunity to choose appropriate tools strategically (MP5).

### Step 1

- Display the equations from the previous activity for all to see (if they are not already visible). Display: $n = 500 \cdot 2^t$ and $p = 100 \cdot 3^t$
- Give students 5 minutes to work in pairs to create graphs of the HeLa cell growth and the parasite growth using the given $t$-values.

### Step 2

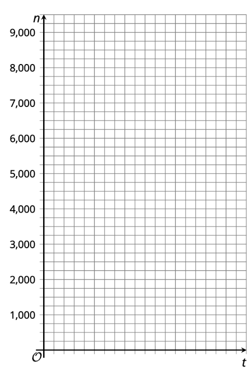
- Use the *Take Turns* routine with *Discussion Supports* to help students describe the meaning of variables in exponential expressions.
- Ask students to arrange themselves in pairs or use visibly random grouping.
  - Invite Partner A to share their observations for the first graph and Partner B to share their observations for the second graph.
  - Provide sentence frames such as “I know the equation and the graph represent the same thing because….” and “I notice that….” Invite the listener to press for additional details referring to features of the graphs, such as the vertical intercept, specific points, and the steepness of the curve. This will help students justify how features of the graph can be used to identify matching equations.

***Advancing Student Thinking***: Students may have trouble graphing the points, particularly finding the appropriate vertical ($n$ or $p$) values. Ask them to find the coordinates of the grid points on the vertical axis and use that to estimate the vertical position of their points. When calculating values by hand, many students may mistakenly write an expression like $100 \cdot 3^2$ as $300^2$. Remind them that the expression $100 \cdot 3^2$ means $100 \cdot 3 \cdot 3$.

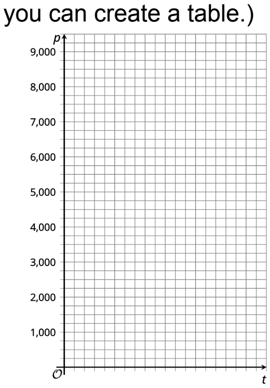
### Student Task Statement

1. Refer back to your work in the table of the previous task. Use that information and the given coordinate planes to graph the following:

   a. Graph $(t, n)$ when $t$ is 0, 1, 2, 3, and 4.




   b. Graph $(t, p)$ when $t$ is 0, 1, 2, 3, and 4. (If you get stuck, you can create a table.)




2. On the graph of $n$, where can you see each number that appears in the equation?
3. On the graph of $p$, where can you see each number that appears in the equation?

## Step 3

- Ask students to identify how each graph shows the rate of change doubling or tripling each day. One way is to identify consecutive $y$ values when $x$ changes by 1, and notice that they double or triple each time. Another way is to look at the vertical change as the horizontal changes by 1, and notice how the vertical change is twice or three times the previous vertical change.
- Make sure students recognize two key takeaways from this activity:
  - The vertical intercept of the graph is the initial HeLa cell population size. It is the size of the population when first measured, or when $t$, the number of days since measurement began, is 0.
  - The growth factor of each population is represented by how quickly the graph increases.

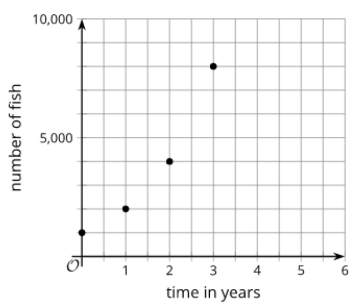
| **DO THE MATH** | **PLANNING NOTES** |
|---|---|
| | |

## Lesson Debrief (*5 minutes*)

The purpose of this lesson is to use equations and graphs to represent situations with quantities that change exponentially. Students learn the term "growth factor" and understand that $b$ in an exponential equation is a growth factor when it is greater than 1.

Choose whether students should first have an opportunity to reflect in their workbooks or talk through these with a partner, and what questions will be prioritized in the full class discussion. Use an example from a classroom activity or a new example like this:

The expression $1000 \cdot 2^t$ represents a fish population after $t$ years. Here is the graph of the yearly fish population. Display for all to see, and ask students where they can see the 1000 and the 2 in the graph.




Ask students:

- "What was the fish population when the scientists first measured it?" (1000)
- "How can you tell from the graph?" (It is the vertical intercept, the number of fish when $t = 0$.)
- "How is the fish population changing from year to year?" (It's doubling.)
- "Why does the expression $1000 \cdot 2^t$ represent the fish population after $t$ years?" (The 1000 is the starting population. Every year it doubles, so we multiply 1000 by 2, $t$ times.)

**PLANNING NOTES**

## Student Lesson Summary and Glossary

A situation where a quantity increases through repeated multiplication by the same amount is called **exponential growth**. This multiplier is called the **growth factor**.

**Exponential growth**: The tendency of something to increase exponentially; that is, the quantity over time can be described by repeated multiplication by a number greater than 1.

**Growth factor:** Quantities changing exponentially can be described as repeated multiplication by a factor, or, in symbols, by the expression $a \cdot b^x$. The multiplier $b$ is called the growth factor when it is greater than 1.

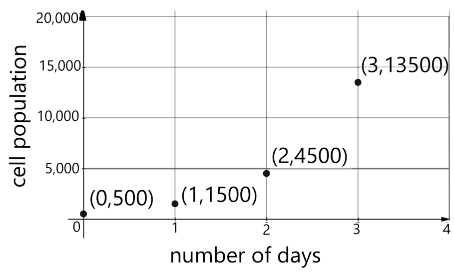
Suppose a population of cells starts at 500 and triples every day. The number of cells each day can be calculated as follows:

| Number of days | Number of cells |
|---|---|
| 0 | $500$ |
| 1 | $1,500$ (or $500 \cdot 3$) |
| 2 | $4,500$ (or $500 \cdot 3 \cdot 3$, or $500 \cdot 3^2$) |
| 3 | $13,500$ (or $500 \cdot 3 \cdot 3 \cdot 3$, or $500 \cdot 3^3$) |
| $d$ | $500 \cdot 3^d$ |

We can see that the number of cells ($p$) is changing exponentially, and that $p$ can be found by multiplying 500 by 3 as many times as the number of days ($d$) since the 500 cells were observed. The growth factor is 3. To model this situation, we can write this equation: $p = 500 \cdot 3^d$.

The equation can be used to find the population on any day: including day 0, when the population was first measured. On day 0, the population is $500 \cdot 3^0$. Since $3^0 = 1$, this is $500 \cdot 1$ or $500$.

Here is a graph of the daily cell population. The point $(0, 500)$ on the graph means that on day 0, the population is $500$.




Each point marked on the graph is 3 times higher on the graph than the previous point. $(1, 1500)$ is 3 times higher than $(0, 500)$, and $(2, 4500)$ is 3 times higher than $(1, 1500)$.

## Cool-down: Mice in the Forest (5 minutes)

**Addressing:** NC.M1.F-LE.5

**Cool-down Guidance:** More Chances
Students will have more opportunities to understand the mathematical ideas in this cool-down, so there is no need to slow down or add additional work to the next lessons. Instead, use the results of this cool-down to provide guidance for what to look for and emphasize over the next several lessons to support students in advancing their current understanding.

### Cool-down

A group of biologists is surveying the mouse population in a forest. The equation $n = 75 \cdot 3^t$ gives the total number of mice, $n$, $t$ years since the survey began. Explain what the numbers 75 and 3 mean in this situation.

**Student Reflection:**

In your own words, how can you use your knowledge about multiplication to help master exponential growth?

## DO THE MATH

## INDIVIDUAL STUDENT DATA

## SUMMARY DATA

## NEXT STEPS

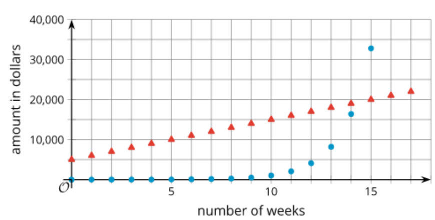
## TEACHER REFLECTION

**What went well in the lesson? What would you do differently next time? What happened today that will influence the planning of future lessons?**

**How did the work of the previous lesson lay the foundation for students to be successful in Activity 1 of this lesson?**

## Practice Problems

1. Bank account A starts with $5,000 and grows by $1,000 each week. Bank account B starts with $1 and doubles each week.

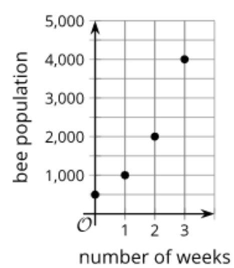



   a. Which account has more money after one week? After two weeks?

   b. Here is a graph showing the two account balances. Which graph corresponds to which situation? Explain how you know.

   c. Given a choice, which of the two accounts would you choose? Explain your reasoning.

2. A bee population is measured each week, and the results are plotted on the graph.




   a. What is the bee population when it is first measured?

   b. Is the bee population growing by the same factor each week? Explain how you know.

   c. What is an equation that models the bee population, $b$, $w$ weeks after it is first measured?

3. A bond is initially bought for \$250. It doubles in value every decade.

| Decades since bond is bought | Dollar value of bond |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| $d$ | |

   a. Complete the table.

   b. How many decades does it take before the bond is worth more than \$10,000?

   c. Write an equation relating $v$, the value of the bond, to $d$, the number of decades since the bond was bought.

4. A sea turtle population $p$ is modeled by the equation $p = 400 \cdot 2^y$ where $y$ is the number of years since the population was first measured.

   a. How many turtles are in the population when it is first measured? Where do you see this in the equation?

   b. Is the population increasing or decreasing? How can you tell from the equation?

   c. When will the turtle population reach 4000? Explain how you know.

5. Which expression is equal to $4^0 \cdot 4^2$?

   a. 0
   b. 1
   c. 16
   d. 64

   (From Unit 6, Lesson 2)

6. Select **all** expressions equivalent to $3^8$.

   a. $8^3$

   b. $\frac{3^{10}}{3^2}$

   c. $3 \cdot 8$

   d. $(3^4)^2$

   e. $(3 \cdot 3)^4$

   f. $\frac{1}{3^{-8}}$

   (From Unit 6, Lesson 2)

7. Without using a calculator, take a guess: Which of the three expressions is the largest: A, B, or C?

| A | B | C |
|---|---|---|
| $9^5 \cdot 9^3$ | $(9^5)^3$ | $9^5 + 9^3$ |

   Now, check your guess using a calculator. Why is the largest expression the largest, based on the exponents?

   (From Unit 6, Lesson 1)

8. Function $F$ is defined so that its output $F(t)$ is the number of followers on a social media account $t$ days after setup of the account.

   a. Explain the meaning of $F(30) = 8,950$ in this situation.

   b. Explain the meaning of $F(0) = 0$.

   c. Write a statement about function $F$ that represents the fact that there were 28,800 followers 110 days after the set up of the account.

   d. Explain the meaning of $t$ in the equation $F(t) = 100,000$.

   (From Unit 5)

9. Match each equation in the first list to an equation in the second list that has the same solution.

   a. $y = \frac{2}{5}x + 2$
   b. $x = -5 - 2.5y$
   c. $y = \frac{10}{5} - 0.4x$
   d. $2x = 10 - 5y$
   e. $-5y = 2x + 10$
   f. $x = 5 - \frac{5}{2}y$

   1. $2x + 5y = 10$
   2. $-2x - 5y = 10$
   3. $-2x + 5y = 10$

   (From Unit 2)

10. Evaluate each expression if $x = 3$.

   a. $2^x$
   b. $x^2$
   c. $1^x$
   d. $x^1$
   e. $(\frac{1}{2})^x$

   (Addressing NC.6.EE.2)[2]

11. Evaluate each expression for the given value of each variable.

   a. $2 + x^3$, $x$ is $3$
   b. $x^2$, $x$ is $\frac{1}{2}$
   c. $3x^2 + y$, $x$ is $5$ and $y$ is $3$
   d. $10y + x^2$, $x$ is $6$ and $y$ is $4$

   (Addressing NC.6.EE.2)[3]

[2] Adapted from IM 6–8 Math https://curriculum.illustrativemathematics.org/MS/index.html, which was originally developed by Open Up Resources and authored by Illustrative Mathematics, and is copyright 2017–2019 by Open Up Resources. It is licensed under the Creative Commons Attribution 4.0 International License (CC BY 4.0). OUR's 6–8 Math Curriculum is available at https://openupresources.org/math-curriculum/. Adaptations and updates to IM 6–8 Math are copyright 2019 by Illustrative Mathematics, and are licensed under the Creative Commons Attribution 4.0 International License (CC BY 4.0).
[3] Adapted from IM 6–8 Math https://curriculum.illustrativemathematics.org/MS/index.html (see above).

# Lesson 6: Understanding Decay

## PREPARATION

| Lesson Goals | Learning Targets |
|---|---|
| • Comprehend that the term "exponential growth" describes a quantity that changes by a growth factor that is greater than 1, and the term "exponential decay" describes a quantity that changes by a growth factor that is less than 1 but greater than 0.<br>• Use only multiplication to represent "decreasing a quantity by some fraction of itself."<br>• Write an expression or an equation $y = a \cdot b^x$ to represent a situation where a quantity decays exponentially. | • I know the meanings of "exponential growth" and "exponential decay."<br>• I can use only multiplication to represent "decreasing a quantity by a fraction of itself."<br>• I can write an expression or equation to represent a quantity that decays exponentially. |

### Lesson Narrative

This lesson continues to examine quantities that change exponentially, this time focusing on a quantity that decays or decreases by a constant factor. Students are alerted that sometimes people use the terms **exponential growth** and **exponential decay** to distinguish between situations where this factor is greater than or less than 1. Additionally, students learn that when the factor is more than 1, it is called the **growth factor**, and when the factor is less than 1 (but still positive), it is called the **decay factor**.

The opening activity encourages students to view a quantity that decreases by a factor of itself using multiplication rather than subtraction. If a computer costs $\$500$ and loses $\frac{1}{5}$ of its value each year, after one year we could write this, in dollars, as $500 - \left(\frac{1}{5}\right) \cdot 500$. But if we write this using multiplication, as $500 \cdot \left(\frac{4}{5}\right)$, then we are in a better position to see that after 2 years, its value in dollars will be $500 \cdot \left(\frac{4}{5}\right) \cdot \left(\frac{4}{5}\right)$ and, after $t$ years, the value will be $500 \cdot \left(\frac{4}{5}\right)^t$. In other words, exponents are a particularly useful way to express repeated loss by a factor of the original amount. Students will carry this understanding into future lessons that deal with repeated percentage change situations.

In the second activity, students apply this idea to write an equation for the value of a car after $t$ years, assuming that the value decreases by the same factor each year.

Using an exponent to express repeated decrease by the same factor is a good example of a generalization based on repeated calculation (MP8). Writing $500 \cdot \left(\frac{4}{5}\right)^t$ expresses the computation of repeatedly decreasing by $\frac{1}{5}$, $t$ times.

**Share some ways you see this lesson connecting to previous lessons in this unit. What connections will you want to make explicit?**

## Focus and Coherence

| Building On | Addressing | Building Towards |
|---|---|---|
| **NC.5.NF.4:** Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction, including mixed numbers.<br>• Use area and length models to multiply two fractions, with the denominators 2, 3, 4.<br>• Explain why multiplying a given number by a fraction greater than 1 results in a product greater than the given number and when multiplying a given number by a fraction less than 1 results in a product smaller than the given number.<br>• Solve one-step word problems involving multiplication of fractions using models to develop the algorithm.<br><br>**NC.7.EE.1:** Apply properties of operations as strategies to:<br>• Add, subtract, and expand linear expressions with rational coefficients.<br>• Factor linear expression with an integer GCF. | **NC.M1.A-CED.2:** Create and graph equations in two variables to represent linear, exponential, and quadratic relationships between quantities.<br><br>**NC.M1.A-SSE.1:** Interpret expressions that represent a quantity in terms of its context.<br>**a.** Identify and interpret parts of a linear, exponential, or quadratic expression, including terms, factors, coefficients, and exponents.<br>**b.** Interpret a linear, exponential, or quadratic expression made of multiple parts as a combination of entities to give meaning to an expression.<br><br>**NC.M1.F-LE.5:** Interpret the parameters $a$ and $b$ in a linear function $f(x) = ax + b$ or an exponential function $g(x) = ab^x$ in terms of a context. | **NC.M1.F-BF.1a:** Build linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two ordered pairs (include reading these from a table). |

## Agenda, Materials, and Preparation

- **Bridge** (*Optional, 5 minutes*)
- **Warm-up** (*5 minutes*)
- **Activity 1** (*10 minutes*)
- **Activity 2** (*15 minutes*)
- **Lesson Debrief** (*5 minutes*)
- **Cool-down** (*5 minutes*)
  - M1.U6.L6 Cool-down (print 1 copy per student)

## LESSON

### Bridge (*Optional, 5 minutes*)

**Building On:** NC.5.NF.4

In this bridge, students will multiply fractions by common factors to prepare to multiply fractions in exponential functions later this lesson. They will have the opportunity to review the procedure for multiplying fractions and understand that fractions with a larger numerator than denominator are greater than 1. Also, in preparing to dig deeper into the growth factors, or bases, in exponential functions, students will explore patterns like:

- Multiplying by a fraction greater than 1 results in a product larger than the other factor.
- Multiplying by a fraction less than 1 results in a product smaller than the other factor.

#### Student Task Statement

Multiply:

1. $2 \cdot 1$
2. $2 \cdot \frac{1}{2}$
3. $2 \cdot \frac{3}{2}$
4. $\frac{5}{8} \cdot 1$
5. $\frac{5}{8} \cdot \frac{1}{2}$
6. $\frac{5}{8} \cdot \frac{3}{2}$
7. What do you notice or wonder about the results when you multiply by $\frac{1}{2}$ compared to when you multiply by $\frac{3}{2}$?

| DO THE MATH | PLANNING NOTES |
|---|---|
| | |

## Warm-up: Two Tables (*5 minutes*)

| |
|---|
| **Instructional Routines:** Notice and Wonder; Collect and Display (MLR2) |
| **Building Towards:** NC.M1.A-CED.2; NC.M1.F-BF.1a |

So far, students have seen patterns and relationships involving mostly whole numbers. This warm-up encourages students to observe exponential growth and decay patterns that involve fractions through the *Notice and Wonder* routine, preparing them for upcoming work.

### Step 1

- Ask students to share what they notice or wonder about the two tables.

### Student Task Statement

What do you notice? What do you wonder?

Table A

| $x$ | $y$ |
|---|---|
| 0 | 2 |
| 1 | 3 |
| 2 | $\frac{9}{2}$ |
| 3 | $\frac{27}{4}$ |
| 4 | $\frac{81}{8}$ |

Table B

| $x$ | $y$ |
|---|---|
| 0 | 1600 |
| 1 | 800 |
| 2 | 400 |
| 3 | 200 |
| 4 | 100 |

### Step 2

- Invite students to share their observations and questions. To encourage more students to participate, after each person shares, ask if others noticed or wondered about the same thing.

- As students share their observations about table B, use the *Collect and Display* routine to capture the terms students use to describe an exponential decay pattern. Leverage this display when students encounter contexts involving patterns of decay in future activities and lessons.
- If not already mentioned by students, highlight how the values in the two tables change from row to row (In table A, the $y$-values in each row increase by a factor of $1.5$ or $\frac{3}{2}$ and in table B, the $y$-value in each row is halved).

| DO THE MATH | PLANNING NOTES |
|---|---|
| | |

## Activity 1: What's Left? (*10 minutes*)

| **Instructional Routine:** Take Turns | | |
|---|---|---|
| **Building On:** NC.7.EE.1 | **Addressing:** NC.M1.A-SSE.1; NC.M1.A-CED.2; NC.M1.F-LE.5 | **Building Towards:** NC.M1.F-BF.1a |

Students have explored quantities that increase by the same factor (e.g., doubling or tripling). Up to this point, that factor has always been greater than 1. In this lesson, they will look at situations where quantities change exponentially, but by a positive factor that is less than 1. This factor, when $0 < b < 1$, is called the decay factor.

Multiplying a quantity by a factor between 0 and 1 means that the quantity is decreasing. Students may naturally think of a decrease as subtraction. For example, a quantity $x$ that loses $\frac{1}{3}$ of its value may be represented as $x - \frac{1}{3}x$. Because exponential relationships are characterized by multiplication by a factor, however, it will be helpful to view this instead as $x \cdot (1 - \frac{1}{3})$, or as *multiplying* $x$ by $\frac{2}{3}$.

### Step 1

- Ask students to close their books or devices. Pose this question: "Diego has $\$100$ and spends $\frac{1}{4}$ of it. How much does he have left?"
- Give students a minute to think about it, and then ask a few students to share their answer and their reasoning.

**RESPONSIVE STRATEGY**

Use color and annotations to illustrate student thinking. As students share their reasoning about Diego's strategy, scribe their thinking on a visible display.

Supports accessibility for: Visual-spatial processing; Conceptual processing

### Step 2

- Ask students to form pairs or use visibly random grouping.

Invite students to *Take Turns* explaining Diego's steps in the task statement.

  – Invite Partner A to begin with the first step, while Partner B listens. Alternate roles for the remaining steps.

  – Encourage students to refer to explicit parts of the expression, such as the operation, specific numbers, the use of parentheses, or the order of the numbers.

  – Encourage students to challenge each other when they disagree.

  – Once they have completed question 1, students will continue to work through the remaining questions with their partner.

M **Monitoring Tip:** As students work, listen for students who make the connection between subtracting a percentage of an amount and multiplying the amount by a related percentage. Ask two or three of these students if they will share their thinking with the class.

***Advancing Student Thinking***: Students may have trouble making the connection between the fraction that is being subtracted and the fraction that remains when the subtraction is rewritten as multiplication. Invite these students to think through the steps in Diego's calculation. What does the number 100 represent? What about the $\frac{1}{4}$? Once students identify the analogous quantities in the second and third questions, encourage them to revisit the subsequent steps of Diego's calculation.

### Student Task Statement

1. Here is one way to think about how much Diego has left after spending $\frac{1}{4}$ of $\$100$. Explain each step.
   - Step 1: $100 - \frac{1}{4} \cdot 100$
   - Step 2: $100(1 - \frac{1}{4})$
   - Step 3: $100 \cdot \frac{3}{4}$
   - Step 4: $\frac{3}{4} \cdot 100$
2. A person makes $\$1,800$ per month, but $\frac{1}{3}$ of that amount goes to her rent. What two numbers can you multiply to find out how much she has after paying her rent?
3. Write an expression that only uses multiplication and that is equivalent to "$x$ reduced by $\frac{1}{8}$ of $x$."

### Step 3

- Ask previously selected students to share their explanations of how a subtraction expression can be written as multiplication of two numbers. Make sure that students understand that a decreasing quantity can be expressed both ways, using subtraction or multiplication.
- If time permits, consider asking: "There were $x$ dogs at the park, and $\frac{1}{5}$ of the dogs were on a leash. How many dogs were not on a leash?" The number of dogs not on a leash could be expressed using subtraction as $x - \frac{1}{5}x$ or using multiplication as $\frac{4}{5}x$.
- Emphasize that it is important to be able to express this situation using multiplication because exponential situations involve quantities changing by a common factor (or being multiplied by a number).

| DO THE MATH | PLANNING NOTES |
| --- | --- |
| | |

## Activity 2: Value of a Vehicle (*15 minutes*)

| | |
|---|---|
| **Instructional Routines:** Co-Craft Questions (MLR5); Poll the Class | |
| **Addressing:** NC.M1.A-CED.2; NC.M1.F-LE.5 | **Building Towards:** NC.M1.F-BF.1a |

In this activity, students examine a situation where a quantity decreases by repeated multiplication by the same factor. They represent the pattern of decrease with an equation and interpret the different parts of the equation in terms of the context (the depreciation of a car). The term and definition of ”decay factor” is introduced to accommodate situations involving decrease and distinguish these situations from situations involving growth. Allow students to use calculators to ensure inclusive participation in the activity.

### Step 1

- Display the first sentence of the task: “Every year after a new car is purchased, it loses $\frac{1}{3}$ of its value. Let’s say that a new car costs $\$18,000$.”

Introduce the term “depreciate.” When the value of something decreases, it depreciates. Use the *Co-Craft Questions* routine to help students make sense of the language of exponential decay.

  - Give students 1–2 minutes to write their own mathematical questions about the situation, then invite students to share their questions with the class.
  - Listen for and amplify any questions about the future value of the car, especially those using language such as “losing value” and “decrease.” This will build student understanding of the language of exponential decay and help ensure students interpret the task correctly.

Ask the class to estimate: Without calculating, what do you think the value of the car will be 4 years after it was purchased as a new car? *Poll the Class* for their estimates.

- Clarify to students that they may complete the table with numerical expressions that show how to find the values rather than with the results of evaluating the expressions.

**Monitoring Tip:** Students may think recursively to begin with, completing each row by using the value from the previous row. This works until the jump to year 6, where they may continue thinking recursively or switch to a more explicit expression. Others may use the explicit expression for the entire table. Help students connect these two ways of thinking in the discussion section.

### Student Task Statement

Every year after a new car is purchased, it loses $\frac{1}{3}$ of its value. Let’s say that a new car costs $\$18,000$.

1. A buyer worries that the car will be worth nothing in three years due to *depreciation*. Do you agree? Explain your reasoning.
2. Write an expression to show how to find the value of the car for each year listed in the table.
3. Write an equation relating the value of the car in dollars, $v$, to the number of years, $t$.
4. Use your equation to find $v$ when $t$ is 0. What does this value of $v$ mean in this situation?

| Year | Expression to calculate the value of car (dollars) | Value of car (dollars) |
|---|---|---|
| 0 | 18,000 | 18,000 |
| 1 | | |
| 2 | | |
| 3 | | |
| 6 | | |
| $t$ | | |

5. A different car loses value at a different rate. The value of this different car in dollars, $d$, after $t$ years can be represented by the equation $d = 10,000 \cdot (\frac{4}{5})^t$.

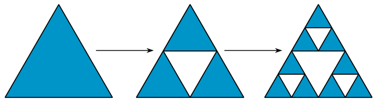
   a. Explain what the numbers $10,000$ and $\frac{4}{5}$ mean in this situation.

   b. Write an expression to show how to find the value of the car for each year listed in the table.

| Year | Expression to calculate the value of car (dollars) | Value of car (dollars) |
|---|---|---|
| 0 | 10,000 | 10,000 |
| 1 | | |
| 2 | | |
| 3 | | |
| 4 | | |
| 5 | | |

## Are You Ready For More?

Start with an equilateral triangle with area 1 square unit, divide it into four congruent pieces as in the figure, and remove the middle one. Then, repeat this process with each of the remaining pieces. Repeat this process over and over for the remaining pieces. The figure shows the first two steps of this construction.



What fraction of the area is removed each time? How much area is removed after the $n$-th step? Use a calculator to find out how much area remains in the triangle after 50 such steps have been taken.

## Step 2

- Ask students to share their equations for the value of the first car ($(v = 18000 \cdot (\frac{2}{3})^t)$. Make sure that students understand the expression for the value of the car after $t$ years came from multiplying by $\frac{2}{3}$ repeatedly ($t$ times). This is where students who computed each row recursively may have run into trouble, so take time here to show how the two methods yield equivalent results.
- Tell students that we still say the value of the car changes exponentially, though sometimes people use the more specific “exponential growth” and “exponential decay” to indicate whether the amount is increasing or decreasing. The multiplier is called the “decay factor” when it is a positive number less than 1, because the result decreases with every iteration.

| DO THE MATH | PLANNING NOTES |
|---|---|
| | |

## Lesson Debrief (*5 minutes*)

The purpose of this lesson is to study quantities that decrease or decay by a constant factor. In this first lesson on the topic, students reason carefully about why multiplying by "one minus" the proportion that decays is both equivalent to and more efficient than using subtraction.

Choose whether students should first have an opportunity to reflect on the following questions in their workbooks or talk through them with a partner. Determine what questions will be prioritized in the full class discussion. Use examples from activities in the lesson or from the new example provided below the Activity 2 questions.

Questions about Activity 2 in this lesson:

- "Why does it make sense to multiply an entry in one row by $\frac{2}{3}$ to get the entry for the next row?" (Each year, only $\frac{2}{3}$ of the car value remains.)
- "Why not write the expression for each year as subtraction, for instance, $18,000 - \frac{1}{3} \cdot 18,000$?" (We could, but writing subtraction can get unwieldy quickly. For year 2, the expression would be $(18,000 - \frac{1}{3} \cdot 18,000) - \frac{1}{3}(18,000 - \frac{1}{3} \cdot 18,000)$. If we calculate the value of each term it would be simpler $(12,000 - \frac{1}{3} \cdot 12,000)$, but it would still not allow us to see a pattern and write a general expression as easily.)
- "Does the $\frac{4}{5}$ in the last equation mean losing $\frac{4}{5}$ of the value each year? Why or why not?" (No. Multiplying by $\frac{4}{5}$ repeatedly means each time only $\frac{4}{5}$ of the value remains, so the car is losing $\frac{1}{5}$ of its value each year.)
- "How could someone looking only at the table in the last question figure out that the decay factor is $\frac{4}{5}$?" (When you divide one car value by the value for the previous year, the quotient is $\frac{4}{5}$.)
- "Will the second car (starting at $\$10,000$ and losing $\frac{1}{5}$ of it each year) ever be worth more than the first car (starting at $\$18,000$ and losing $\frac{1}{3}$ of it each year)? Explain." (Yes. After 4 years, the value of the second car will be $\$4,096$, while the value of the first car will be $\$3,555.56$.)

Additional Example: Suppose a digital camera worth $\$400$ loses $\frac{1}{3}$ of its value each year.

- "How can we express the value of the camera, in dollars, after one year? After two years?" ($400 \cdot \frac{2}{3}^{(1)}$; $400 \cdot \frac{2}{3}^{(2)}$)
- "Why might it make sense to use only multiplication (instead of subtraction and multiplication) to show the value of the camera over time?" (After many years, there might be too many numbers to subtract easily.)
- "When we use only multiplication, why doesn't the $\frac{1}{3}$ show up in the expression?" (We are not taking away $\frac{1}{3}$ of the value; rather, we are keeping $\frac{2}{3}$ of the value.)
- "What is the value of the camera, in dollars, after $t$ years?" ($400 \cdot \frac{2}{3}^{t}$)

**PLANNING NOTES**

### Student Lesson Summary and Glossary

Sometimes a quantity grows by the same factor at regular intervals. For example, a population doubles every year. Sometimes a quantity *decreases* by the same factor at regular intervals. For example, a car might lose one third of its value every year. When an object loses value over time, we say that it **depreciates**.

**Depreciate**: To lose value over time. For example, a car is worth less money the older it is.

Let's look at a situation where a quantity decreases by the same factor at regular intervals. Suppose a bacteria population starts at $100,000$ and $\frac{1}{4}$ of the population dies each day. The population one day later is $100,000 - \frac{1}{4} \cdot 100,000$, which can be written as $100,000(1 - \frac{1}{4})$. The population after one day is $\frac{3}{4}$ of $100,000$: or $75,000$. The population after two days is $\frac{3}{4} \cdot 75,000$. Here are some further values for the bacteria population

| Number of days | Bacteria population |
|---|---|
| 0 | $100,000$ |
| 1 | $75,000$ (or $100,000 \cdot \frac{3}{4}$) |
| 2 | $56,250$ (or $100,000 \cdot \frac{3}{4} \cdot \frac{3}{4}$, or $100,000 \cdot (\frac{3}{4})^2$) |
| 3 | about $42,188$ (or $100,000 \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4}$, or $100,000 \cdot (\frac{3}{4})^3$) |

In general, $d$ days after the bacteria population was $100,000$, the population $p$ is given by the equation: $p = 100,000 \cdot (\frac{3}{4})^d$, with one factor of $\frac{3}{4}$ for each day.

Situations with quantities that decrease exponentially are described as **exponential decay**. The multiplier ($\frac{3}{4}$ in this case) is called the **decay factor**.

**Exponential decay:** The tendency of a quantity to decrease exponentially: that is, the quantity remaining over time can be described by repeated multiplication by a number between 0 and 1.

**Decay factor:** Quantities changing exponentially can be described as repeated multiplication by a factor, or, in symbols, by the expression $a * b^x$. The multiplier *b* is referred to as the decay factor when this number is between 0 and 1.

## Cool-down: The Depreciating Phone (*5 minutes*)

**Addressing:** NC.M1.A-CED.2

**Cool-down Guidance:** More Chances
Students will have more opportunities to understand the mathematical ideas in this cool-down, so there is no need to slow down or add additional work to the next lessons. Instead, use the results of this cool-down to provide guidance for what to look for and emphasize over the next several lessons to support students in advancing their current understanding.

### Cool-down

Suppose that a phone that originally sold for $\$800$ loses $\frac{3}{5}$ of its value each year after it is released.

1. After 2 years, how much is the phone worth?
2. Write an equation for the value of the phone, $p$, $t$ years after it is released.

**Student Reflection:**

What was the best question your teacher asked you today that helped with your thinking and/or learning?

## DO THE MATH

## INDIVIDUAL STUDENT DATA

## SUMMARY DATA

## NEXT STEPS

## TEACHER REFLECTION

**What went well in the lesson? What would you do differently next time? What happened today that will influence the planning of future lessons?**

**Think of the ways you engaged students in discourse today. What was the best question you asked students today? What did you notice about the responses you received?**

## Practice Problems

1. A new bicycle sells for $\$300$. It is on sale for $\frac{1}{4}$ off the regular price. Select **all** the expressions that represent the sale price of the bicycle in dollars.

    a. $300 \cdot \frac{1}{4}$
    b. $300 \cdot \frac{3}{4}$
    c. $300 \cdot (1 - \frac{1}{4})$
    d. $300 - \frac{1}{4}$
    e. $300 - \frac{1}{4} \cdot 300$

2. A computer costs $\$800$. It loses $\frac{1}{4}$ of its value every year after it is purchased.

    a. Complete the table to show the value of the computer at the listed times.

    b. Write an equation representing the value, $v$, of the computer, $t$ years after it is purchased.

    c. Use your equation to find $v$ when $t$ is 5. What does this value of $v$ mean?

| Time (years) | Value of computer (dollars) |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| $t$ | |

3. A piece of paper is folded into thirds multiple times. The area, $A$, of the piece of paper in square inches, after $n$ folds, is $A = 90 \cdot (\frac{1}{3})^n$.

    a. What is the value of $A$ when $n = 0$? What does this mean in this situation?

    b. How many folds are needed before the area is less than 1 square inch?

    c. The area of another piece of paper in square inches, after $n$ folds, is given by $B = 100 \cdot (\frac{1}{2})^n$. What do the numbers $100$ and $\frac{1}{2}$ mean in this situation?

4. At the beginning of April, a colony of ants has a population of 5,000.

   a. The colony decreases by $\frac{1}{5}$ during April. Write an expression for the ant population at the end of April.

   b. During May, the colony decreases again by $\frac{1}{5}$ of its size. Write an expression for the ant population at the end of May.

   c. The colony continues to decrease by $\frac{1}{5}$ of its size each month. Write an expression for the ant population after 6 months.

5. Lin starts with 13 mystery novels. Each month, she gets two more. Select **all** expressions that represent the total number of Lin's mystery novels after 3 months.

   a. $13+2+2+2$
   b. $13 \cdot 2 \cdot 2 \cdot 2$
   c. $13 \cdot 8$
   d. $13+6$
   e. $19$

6. An odometer is the part of a car's dashboard that shows the number of miles a car has traveled in its lifetime. Before a road trip, a car's odometer reads 15,000 miles. During the trip, the car travels 65 miles per hour.

   a. Complete the table.

   b. What do you notice about the differences of the odometer readings each hour?

   c. If the odometer reads $n$ miles at a particular hour, what will it read one hour later?

| Duration of trip (hours) | Odometer reading (miles) |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |

   (From Unit 6, Lesson 4)

7. Rewrite and simplify the following expressions so all exponents are positive:

   a. $\frac{5x^4y^{-2}z}{15x^{-3}yz^4}$

   b. $(\frac{2}{3}a^4b^2c^{-4})^2$

   (From Unit 6, Lesson 2)

8. A function multiplies its input by $\frac{3}{4}$ then adds 7 to get its output. Use function notation to represent this function.

   (From Unit 5)

9. A function is defined by the equation $f(x) = 2x - 5$.

   a. What is $f(0)$?

   b. What is $f(\frac{1}{2})$?

   c. What is $f(100)$?

   d. What is $x$ when $f(x) = 9$?

   (From Unit 5)

10. A group of students is collecting 16 oz and 28 oz jars of peanut butter to donate to a food bank. At the end of the collection period, they donated 1,876 oz of peanut butter and a total of 82 jars of peanut butter to the food bank.

    a. Write a system of equations that represents the constraints in this situation. Be sure to specify the variables that you use.

    b. How many 16 oz jars and how many 28 oz jars of peanut butter were donated to the food bank? Explain or show how you know.

    (From Unit 2)

# Lesson 7: Representing Exponential Decay

## PREPARATION

| Lesson Goals | Learning Targets |
|---|---|
| • Calculate a decay factor using points on a graph that represents exponential decay.<br>• Graph equations that represent quantities that change by a factor between 0 and 1.<br>• Interpret equations and graphs that represent exponential decay situations. | • I can find a decay factor from a graph and write an equation to represent exponential decay.<br>• I can graph equations that represent quantities that change by a factor between 0 and 1.<br>• I can explain the meanings of $a$ and $b$ in an equation that represents exponential decay and is written as $y = a \cdot b^x$. |

### Lesson Narrative

In this lesson, students examine more situations with quantities that decrease exponentially. They work from an equation to a graph and from a graph to an equation. In both cases, they interpret the different parts of their equation in terms of the situation and use the graph to answer questions.

Like many activities in this unit, the equations and graphs represent actual quantities (the area covered by algae and the amount of insulin in a person's body) and are to be interpreted in context (MP2). The activities also use a discrete graph to answer questions about quantities that vary continuously with time, but the concept of appropriate use of when continuous graphs are appropriate is not addressed until later lessons. While technology isn't required for this lesson, there are opportunities for students to choose to use appropriate technology to solve problems.

**What strategies or representations do you anticipate students might use in this lesson?**

### Focus and Coherence

| Building On | Addressing | Building Towards |
|---|---|---|
| **NC.8.EE.1:** Develop and apply the properties of integer exponents to generate equivalent numerical expressions. | **NC.M1.A-CED.2:** Create and graph equations in two variables to represent linear, exponential, and quadratic relationships between quantities.<br><br>**NC.M1.F-IF.7:** Analyze linear, exponential, and quadratic functions by generating different representations, by hand in simple cases and using technology for more complicated cases, to show key features, including: domain and range; rate of change; intercepts; intervals where the function is increasing, decreasing, positive, or negative; maximums and minimums; and end behavior.<br><br>**NC.M1.F-LE.5:** Interpret the parameters $a$ and $b$ in a linear function $f(x) = ax + b$ or an exponential function $g(x) = ab^x$ in terms of a context. | **NC.M1.F-BF.1a:** Build linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two ordered pairs (include reading these from a table). |

## Agenda, Materials, and Preparation

- **Bridge** (*Optional, 5 minutes*)
- **Warm-up** (*5 minutes*)
- **Activity 1** (*10 minutes*)
  - Algae bloom visual (print or display for students to reference)
  - Technology is suggested for this activity: Acquire devices that can access Desmos (recommended) or other graphing technology. It is ideal if each student has their own device.
- **Activity 2** (*15 minutes*)
- **Lesson Debrief** (*5 minutes*)
- **Cool-down** (*5 minutes*)
  - M1.U6.L7 Cool-down (print 1 copy per student)

## LESSON

### Bridge (*Optional, 5 minutes*)

**Building On:** NC.8.EE.1

The purpose of this bridge is to provide additional practice for students to connect exponents with repeated multiplication.

**Student Task Statement**

Write each expression using an exponent.[1]

1. $1 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7$
2. $1 \cdot \left(\frac{4}{5}\right) \cdot \left(\frac{4}{5}\right) \cdot \left(\frac{4}{5}\right) \cdot \left(\frac{4}{5}\right) \cdot \left(\frac{4}{5}\right)$
3. $1 \cdot (9.3) \cdot (9.3) \cdot (9.3) \cdot (9.3) \cdot (9.3) \cdot (9.3) \cdot (9.3) \cdot (9.3)$
4. The number of coins Jada will have on the eighth day if she starts with one coin and the number of coins doubles every day. (She has two coins on the first day of the doubling.)

| DO THE MATH | PLANNING NOTES |
| --- | --- |
| | |

[1] Adapted from IM 6–8 Math https://curriculum.illustrativemathematics.org/MS/index.html, which was originally developed by Open Up Resources and authored by Illustrative Mathematics, and is copyright 2017–2019 by Open Up Resources. It is licensed under the Creative Commons Attribution 4.0 International License (CC BY 4.0). OUR's 6–8 Math Curriculum is available at https://openupresources.org/math-curriculum/. Adaptations and updates to IM 6–8 Math are copyright 2019 by Illustrative Mathematics, and are licensed under the Creative Commons Attribution 4.0 International License (CC BY 4.0).

## Warm-up: Two Other Tables (*5 minutes*)

**Building Towards:** NC.M1.A-CED.2; NC.M1.F-BF.1a

This warm-up asks students not only to identify that a quantity is changing linearly or exponentially but also to identify a term when the preceding value is not given.

### Step 1

- Ask students to find patterns and complete the tables.

***Advancing Student Thinking*:** For the last entry in each table, students may struggle to find the actual value to write in the table. Explain that an expression representing the value is all that's needed. If necessary, prompt students to show repeated multiplication in equivalent expressions for the values of $y$ given in the table, such as $2.5(4)^1$, $2.5(4)^2$, $2.5(4)^3$, in order to see the structure for the expression needed.

### Student Task Statement

Use the patterns you notice to complete the tables. Show your reasoning.

Table A

| $x$ | 0 | 1 | 2 | 3 | 4 | 25 |
|---|---|---|---|---|---|---|
| $y$ | 2.5 | 10 | 17.5 | 25 | | |

Table B

| $x$ | 0 | 1 | 2 | 3 | 4 | 25 |
|---|---|---|---|---|---|---|
| $y$ | 2.5 | 10 | 40 | 160 | | |

### Step 2

- Facilitate a whole-class discussion focused on how students used the patterns in the tables to generate a $y$-value for the non-consecutive values of $x$.
  - Make sure students are able to express the pattern of repeated addition in the first table using an expression like $2.5 + 7.5x$, and that they can express the pattern of repeated multiplication in the second table using an exponential expression like $(2.5) \cdot 4^x$.
  - Highlight the words "linear" and "exponential" to describe how the quantities are changing suggested by the tables.
- In Lesson 3, students calculated the factor from the previous year by dividing the current value by the previous value. Highlight that the same method can be used to find the factor in table B, even though they can do it easily by looking at 10 to 40. Students will need different methods for finding the factor in subsequent activities.

**DO THE MATH**

**PLANNING NOTES**

## Activity 1: The Algae Bloom (*10 minutes*)

| | |
|---|---|
| **Instructional Routines:** Graph It; Critique, Correct, Clarify (MLR3) | |
| **Addressing:** NC.M1.A-CED.2; NC.M1.F-IF.7; NC.M1.F-LE.5 | |

In this *Graph It* activity, students focus on producing and interpreting a graph representing a quantity that decays exponentially. They can then use the graph to make some estimates about the context, which is about algae control. After students create their graph, use the activity synthesis to encourage students to interpret the meaning of input values that are not whole numbers.

Note that the axes on the given graph aren't scaled. Students will need to make sure that the vertical intercept can be plotted and that it is not too close to 0. If students do the work to decide how to scale the axes, they are creating a coherent representation of the relationship (MP2).

Making graphing technology available gives students an opportunity to choose appropriate tools strategically (MP5).

### Step 1

- To introduce the context and keep students thinking about exponential growth and decay, consider presenting a riddle:
  - A certain area of the pond is covered in algae. Each day, the algae-covered area doubles. If it takes 24 days for the algae to completely cover the pond, how many days did it take to cover half of the pond?
  - Given the word "half," students may be inclined to immediately say that it takes 12 days. But if they ponder the situation a bit, they should recognize that it takes 23 days to cover half the pond, because it only takes one more day or one more doubling to cover the whole pond.
- Display the algae bloom visual.
- Provide students with quiet think time to read the questions and respond.

**RESPONSIVE STRATEGY**

Activate or supply background knowledge. Provide students with access to a blank table of values. Invite students to complete the table for when t is 0, 1, 2, 3, and 4, and then discuss how to use this information to choose an appropriate scale for the axes.

Supports accessibility for: Visual-spatial processing; Organization

***Advancing Student Thinking***: Students may find it challenging to choose a scale for the axes in a way that helps them plot the points and see a pattern. If they are still struggling to choose a scale for the axes after a few minutes, ask students to think about the greatest and least vertical coordinates they need to show on the graph, and what the height of each rectangle on the grid should be to show these values. In addition, if they get stuck plotting points, suggest that they first make a table of values.

Additionally, students may be unsure how to interpret $t = -1$ in the last question. Ask these students what a $t$-value of 1 means, a $t$-value of 0, then ask them what they think a $t$-value of -1 should mean. Many students will be satisfied after correctly calculating the value of A when $t = -1$. Ask these students to reread the second sentence of the task. What happened at time zero?

### Student Task Statement

In order to control an algae bloom in a lake, scientists introduce some treatment products.

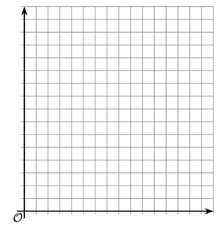
Once the treatment begins, the area covered by algae $A$, in square yards, is given by the equation $A = 240 \cdot (\frac{1}{3})^t$. Time, $t$, is measured in weeks.

1. In the equation, what does the 240 tell us about the algae? What does the $\frac{1}{3}$ tell us?

2. Create a graph to represent $A = 240 \cdot \left(\frac{1}{3}\right)^t$ when $t$ is 0, 1, 2, 3, and 4. Think carefully about how you choose the scale for the axes. If you get stuck, consider creating a table of values.

3. Approximately how many square yards will the algae cover after 2.5 weeks? Explain your reasoning.

4. Use the rule to evaluate $A$ when $t$ is -1. Do the values make sense in this situation? Explain.



| $t$ | $A$ |
|---|---|
| | |
| | |
| | |
| | |
| | |
| | |
| | |

## Are You Ready For More?

The scientists estimate that to keep the algae bloom from spreading after the treatment concludes, they will need to reduce the area covered to under one square foot. How many weeks should they run the treatment in order to achieve this?

## Step 2

- Once students have created reasonable graphs, use the *Critique, Correct, Clarify* routine to focus on reasoning and communicating about what is happening to the algae over time.
  - Display this first draft statement for students to analyze and improve: “After 7 weeks, the algae will disappear because it's shrinking by a third each week so by then it will be gone."
  - Give students 1 minute of individual think time to identify parts of the statement that are unclear, incomplete, and/or incorrect and then invite two or three students to share their ideas. Annotate the statement as students share, to indicate parts that need improvement.
  - Give students 1 minute in pairs or individually to revise the statement to make it more clear, more correct, and more complete. As students work, monitor for useful language moves, phrases, and words.
  - Ask one student to read their improved draft aloud, and scribe their words for all to see as they read aloud. Invite collective editing from the class while scribing to generate a more refined third draft for all students to record.
  - Sample improved drafts:
    - “After 7 weeks, the algae will almost disappear if we keep multiplying by $\frac{1}{3}$ because the area will keep getting smaller, but never reach 0.”
    - “After 7 weeks, the algae will take up about $\frac{1}{9}$ of a square yard or 1 square foot covered by algae, which is 0 square yards if we round to the nearest square yard.”

## DO THE MATH

## PLANNING NOTES

## Activity 2: Insulin in the Body (*15 minutes*)

| **Instructional Routines:** Aspects of Mathematical Modeling; Notice and Wonder; Three Reads (MLR6); Compare and Connect (MLR7) | |
|---|---|
| **Addressing:** NC.M1.A-CED.2; NC.M1.F-IF.7 | **Building Towards:** NC.M1.F-BF.1a |

Here, students make sense of a graph representing a situation characterized by exponential decay. They justify why the amount of insulin in a patient's body could change exponentially and use the graph to answer questions about the situation. The numbers, chosen explicitly to provide a realistic model of the insulin context, are more complex than what students have seen so far. They will need to apply their understanding both to identify the decay factor and to write an expression showing how much insulin remains after $m$ minutes.

In this activity, students are building skills that will help them in mathematical modeling (MP4). They don't decide which model to use, but use *Aspects of Mathematical Modeling* as they figure out how to justify that an exponential model is a good model for the data. Also, they are prompted to construct an equation to represent the model in a scaffolded way.

Allow students to use calculators to ensure inclusive participation in the activity.

### Step 1

- Explain that insulin is a hormone that is needed to process glucose (sugar) in the body. For people with diabetes, a disorder in which the body cannot process glucose properly or does not produce enough insulin, injections of insulin are sometimes needed. Once in the bloodstream, insulin breaks down fairly rapidly, so additional injections are needed over the course of a day to control the person's glucose.

Display the description of the situation and the graph for all to see. Ask students to observe the graph and be prepared to share something they *Notice and Wonder*. Give students a moment to share their observations and questions with a partner.

Use the *Three Reads* routine to support reading comprehension without solving for students. Do not display the questions in the task statement until after the second read.

  – Use the first read to orient students to the situation and identify contextual information that may be unfamiliar to students. Ask students to describe what the situation is about without using numbers (a diabetic patient who receives insulin).

  – Use the second read to identify quantities and relationships. Ask students what can be counted or measured, without focusing on the values. Listen for, and amplify, the important quantities that vary in relation to each other in this situation: amount of insulin and time in minutes.

  – After the third read, ask students to brainstorm possible strategies to answer the first two questions.

**RESPONSIVE STRATEGY**

During the second read, annotate the graph (scale, axis titles, ordered pairs labeled with context) to support students with the conceptual understanding of the task.

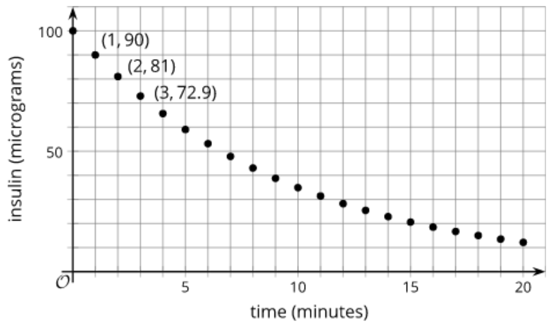
Supports accessibility for: Conceptual processing

- Allow students to work in pairs or independently on the task.

***Advancing Student Thinking***: Students may notice that the insulin values decrease by $\frac{1}{10}$ each minute and say, in response to the first two questions, that $\frac{1}{10}$ of the original insulin remains after each minute. Ask them what $\frac{1}{10}$ of 10 mg is. Then ask them if that is the amount of insulin that remains after 1 minute. If needed, ask them what fraction of 10 mg is the amount of insulin that remains after 1 minute. Is that the same as the fraction of 9 mg of insulin that remains after 2 minutes?

### Student Task Statement

A patient who is diabetic receives 100 micrograms of insulin. The graph shows the amount of insulin, in micrograms, remaining in his bloodstream over time, in minutes.




1. Scientists have found that the amount of insulin in a patient's body changes exponentially. How can you check if the graph supports the scientists' claim?
2. How much insulin broke down in the first minute? What fraction of the original insulin is that?
3. How much insulin broke down in the second minute? What fraction is that of the amount one minute earlier?
4. What fraction of insulin remains in the bloodstream for each minute that passes? Explain your reasoning.
5. Complete the table to show the predicted amount of insulin 4 and 5 minutes after injection.

| Time after injection (minutes) | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| **Insulin in the bloodstream (micrograms)** | 100 | 90 | 81 | 72.9 | | |

6. Describe how you would find how many micrograms of insulin remain in his bloodstream after 10 minutes. After $m$ minutes?

### Step 2

- Prepare for a *Compare and Connect* routine by displaying the context and the graph with space to annotate during the whole-class discussion. As students share their thinking, show connections between representations using the visual display. Include additional language that students use in their explanations.
- Invite students to share their responses with the class. Ask students:
  - "How did you find the decay factor, and where do we see the decay factor in the context and in our mathematical work"? (I divided 90 by 100, or I noticed that 90 is $\frac{9}{10}$ of 100.)
  - "How can we use an equation to express the milligrams of insulin in the body ($I$) after $m$ minutes using an equation?" ($I = 100 \cdot (\frac{9}{10})^m$)
  - "Where can we see the 100 mg in the graph?" (the vertical intercept)
  - "What about the $\frac{9}{10}$?" (The coordinates tell us that $\frac{1}{10}$ of insulin gets removed each minute, so $\frac{9}{10}$ of it, which is most of it, is left in the body every minute. The amount of insulin decays relatively slowly, which shows in the points dropping gradually.)
- Also consider asking:
  - "Will there be any insulin remaining in the bloodstream an hour after the injection?"
  - "If so, when will the body completely run out of insulin?"

The last two questions are difficult to answer. The mathematical model will never reach 0 because a positive quantity multiplied by $\frac{9}{10}$ is always positive. Practically speaking, however, the insulin given to the patient will eventually all disappear, unless a new dose is injected. The model also doesn't account for any insulin that the body produces naturally. This is a good opportunity to remind students that mathematical models are simplified descriptions of reality.

## DO THE MATH

## PLANNING NOTES

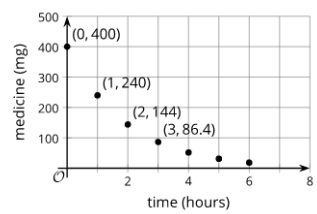
## Lesson Debrief (*5 minutes*)

The purpose of this lesson is to have students explore several science-related contexts that present as exponential decay patterns. They practice connecting equations to graphs and graphs to equations as they reason about problems stemming from these contexts.

Choose whether students should first have an opportunity to reflect on the following questions in their workbooks or talk through them with a partner. Determine what questions will be prioritized in the full class discussion.

Use the following new example to stimulate conversation.

Here is a graph showing the amount, in mg, of a medicine in a person's body at some different times, measured in hours, after taking some medicine. Every hour, $\frac{2}{5}$ of the medicine is broken down by the body.



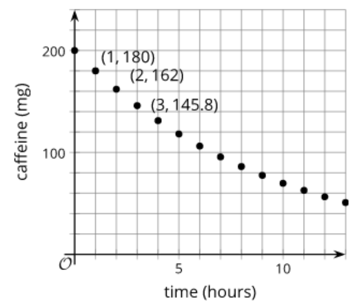

Discuss questions such as:

- "What is the vertical intercept and what does it mean in this context?" (400. It is the milligrams of medicine in the person's body right after taking the pill.)
- "How can you tell from the graph that $\frac{2}{5}$ of the medicine is broken down after one hour?" (The amount of medicine after 1 hour dropped by 160 mg, and 160 is $\frac{2}{5}$ of 400. Based on what was emphasized in Activity 2, students may look at current/previous = 240/400 and find that $\frac{3}{5}$ remains, meaning $\frac{2}{5}$ is used. )
- "How can you tell from the graph that $\frac{3}{5}$ remained after each hour?" (240 is $\frac{3}{5}$ of 400, and 144 is $\frac{3}{5}$ of 240, and so on.)
- "What is an equation representing the amount of medicine in mg, $m$, $t$ hours after taking the medicine?" ($m = 400(\frac{3}{5})^t$)
- "Why does it make sense to write $(\frac{3}{5})^t$?" ($\frac{3}{5}$of the medicine remained with the passing every hour, so for $t$ hours, we multiply the 400 by $\frac{3}{5}$, $t$ times.)
- What would be the interpretation of $t = -1$? Can we use the equation to find the amount of medicine in the body when $t = -1$? ($t = -1$ means one hour before the patient took the medicine. We can't use the equation to find this amount because the patient did not take the medicine until $t = 0$.)

## PLANNING NOTES

### Student Lesson Summary and Glossary

Here is a graph showing the amount of caffeine in a person's body, measured in milligrams, over a period of time, measured in hours. We are told that the amount of caffeine in the person's body changes exponentially.




The graph includes the point $(0, 200)$. This means that there were 200 milligrams of caffeine in the person's body when it was initially measured. The point $(1, 180)$ tells us there were 180 milligrams of caffeine 1 hour later. Between 6 and 7 hours after the initial measurement, the amount of caffeine in the body fell below 100 milligrams.

We can use the graph to find out what fraction of caffeine remains in the body each hour. Notice that $\frac{180}{200} = \frac{9}{10}$ and $\frac{162}{180} = \frac{9}{10}$. As each hour passes, the amount of caffeine that stays in the body is multiplied by a factor of $\frac{9}{10}$.

If $y$ is the amount of caffeine, in milligrams, and $t$ is time, in hours, then this situation is modeled by the equation: $y = 200 \cdot \left(\frac{9}{10}\right)^t$.

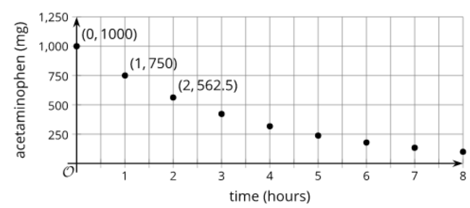
## Cool-down: Acetaminophen (*5 minutes*)

| **Addressing:** NC.M1.A-CED.2 | **Building Towards:** NC.M1.F-BF.1a |
|---|---|
| **Cool-down Guidance:** Points to Emphasize<br>If students are still struggling with writing equations to represent exponential growth or decay, highlight a few examples from cool-downs and use practice problems from this lesson to provide further opportunities to examine the meaning of $a$ and $b$ in the expression $a \cdot b^x$ and to practice writing equations that represent exponential decay. | |

### Cool-down

Acetaminophen is a common pain reliever and fever reducer. Here is a graph showing the amount of acetaminophen, in milligrams, in an adult's body at different times after they take a normal dose.

1. What is the vertical intercept? What does it mean in this situation?
2. What fraction of the medicine remained after 1 hour?
3. Write an equation that represents the amount of medicine, $a$, after $t$ hours.




**Student Reflection:**

True or false: The use of science-related topics to help learn about exponential decay helped me with my understanding. Explain your reasoning.

### DO THE MATH

## INDIVIDUAL STUDENT DATA

## SUMMARY DATA

## NEXT STEPS

## TEACHER REFLECTION

**What went well in the lesson? What would you do differently next time? What happened today that will influence the planning of future lessons?**

**Which types of practice problems from this unit or previous units have been used to support students in building an understanding of exponential growth and exponential decay? How will you give students opportunities to continue to work with those types of problems?**

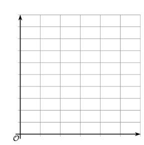
## Practice Problems

1. A population $p$ of migrating butterflies satisfies the equation $p = 100,000 \cdot \left(\frac{4}{5}\right)^w$ where $w$ is the number of weeks since they began their migration.

   a. Complete the table with the population after different numbers of weeks.

| $w$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $p$ | | | | | |

   b. Graph the butterfly population. Think carefully about how to choose a scale for the axes.

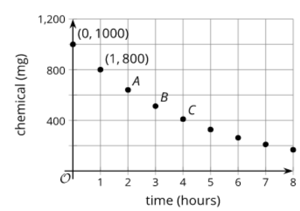


   c. What is the vertical intercept of the graph? What does it tell you about the butterfly population?

   d. About when does the butterfly population reach 50,000?

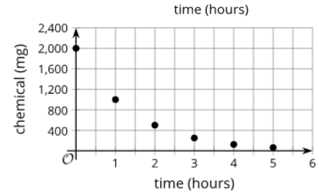
2. The graph shows the amount of a chemical in a water sample. It is decreasing exponentially.

   Find the coordinates of the points labeled $A$, $B$, and $C$. Explain your reasoning.




3. The graph shows the amount of a chemical in a patient's body at different times measured in hours since the levels were first checked.

   Could the amount of this chemical in the patient be decaying exponentially? Explain how you know.




4. The height of a plant is 7 mm. It doubles each week. Select **all** expressions that represent the height of the plant, in mm, after 4 weeks.

   a. $7 + 4 \cdot 2$
   b. $7 \cdot 2^4$
   c. $2 + 7^4$
   d. $7 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
   e. $7 \cdot 2 \cdot 4$

5. At the beginning of January, 300 people have read a new book. The number of people who have read the book doubles each month.

| Number of months since the beginning of January | Number of people who have read the book |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| 4 | |

   a. Use this information to complete the table.

   b. What do you notice about the difference in the number of people who have read the book from month to month?

   c. What do you notice about the factor by which the number of people changes each month?

   d. If $n$ people have read the book one month, how many people will have read the book the following month?

6. The number of Netflix subscribers in Latin America has increased a lot in recent years.[2] The number of paid subscribers from 2018–2020 was:

| **Year** | 2018 | 2019 | 2020 |
|---|---|---|---|
| **Number of Subscribers (in millions)** | 26.08 | 31.42 | 37.54 |

Noah and Elena agree that the number of subscribers is probably increasing exponentially—everyone loves Netflix! Noah says the growth factor is about 20%, or 0.2, but Elena disagrees and says the growth factor is about 1.2. Who do you agree with? Explain your reasoning.

(From Unit 6, Lesson 5)

7. Researchers at the University of North Carolina are studying the spread of diseases. For a new bacterial disease, they are able to isolate one cell, and they watch as it divides into three cells over the first hour. The number of cells grows exponentially until there are $3^5$ cells after 5 hours. The researchers are nervous about the constant growth, and they keep watching the bacteria grow. Over the next 4 hours, the number of cells continues to grow, multiplying the previous total by $3^4$.

Express the total number of cells after 9 hours first as an exponential expression and then as a whole number.

(From Unit 6, Lesson 1)

8. Tyler creates a scatter plot that displays the relationship between the grams of food a hamster eats, $x$, and the total number of rotations that the hamster's wheel makes, $y$. They create a line of best fit and find that the residual for the point $(1.2, 1364)$ is 117. Interpret the meaning of 117 in the context of the problem.

(From Unit 4)

9. Solve each system of equations.

   a. $\begin{cases} x + y = 2 \\ -3x - y = 5 \end{cases}$

   b. $\begin{cases} \frac{1}{2}x + 2y = -13 \\ x - 4y = 8 \end{cases}$

(From Unit 3)

10. $8^{12} \cdot 8^{-20} = 8^x$

What value of $x$ makes the equation true?

(Addressing NC.8.EE.1)

[2] Dean, B. (2021, February 25). *Netflix subscriber and growth statistics: How many people watch Netflix in 2021?* Backlinko. https://backlinko.com/netflix-users#netflix-payed-suscriber-growth

# Lesson 8: Analyzing Graphs

## PREPARATION

| Lesson Goals | Learning Targets |
|---|---|
| • Determine whether situations are characterized by exponential growth or by exponential decay given descriptions and graphs.<br>• Use graphs to compare and contrast situations that involve exponential decay.<br>• Use information from a graph to write an equation that represents exponential decay. | • I can tell whether a situation involves exponential growth or exponential decay based on a description or a graph.<br>• I can use graphs to compare and contrast situations that involve exponential decay.<br>• I can use information from a graph to write an equation that represents exponential decay. |

### Lesson Narrative

In this lesson, students continue to examine situations characterized by exponential decay. The emphasis here is on analyzing graphs representing such situations. Students work across representations from graphs to equations and from verbal descriptions to graphs. In addition to interpreting mathematical representations in context (MP2), students also think carefully about how the numbers used in place of $a$ and $b$ in an expression of the form $a \cdot b^x$ influence the graph of the equation $y = a \cdot b^x$ (MP7).

**What are you excited for your students to be able to do after this lesson?**

### Focus and Coherence

| Building On | Addressing | Building Towards |
|---|---|---|
| **NC.6.NS.7:** Understand ordering of rational numbers.<br>**a.** Interpret statements of inequality as statements about the relative position of two numbers on a number line diagram.<br>**b**. Write, interpret, and explain statements of order for rational numbers in real-world contexts.<br><br>**NC.8.EE.1:** Develop and apply the properties of integer exponents to generate equivalent numerical expressions.<br><br>**NC.8.F.4:** Analyze functions that model linear relationships.<br>• Understand that a linear relationship can be generalized by $y = mx + b$.<br><br>(*continued*) | **NC.M1.A-CED.2:** Create and graph equations in two variables to represent linear, exponential, and quadratic relationships between quantities.<br><br>**NC.M1.F-IF.7:** Analyze linear, exponential, and quadratic functions by generating different representations, by hand in simple cases and using technology for more | **NC.M1.F-BF.1a:** Build linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two ordered pairs (include reading these from a table).<br><br>**NC.M1.F-IF.9:** Compare key features of two functions (linear, |

| | | |
|---|---|---|
| • Write an equation in slope-intercept form to model a linear relationship by determining the rate of change and the initial value, given at least two $(x, y)$ values or a graph.<br>• Construct a graph of a linear relationship given an equation in slope-intercept form.<br>• Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of the slope and $y$-intercept of its graph or a table of values.<br><br>**NC.M1.F-IF.4:** Interpret key features of graphs, tables, and verbal descriptions in context to describe functions that arise in applications relating two quantities, including: intercepts; intervals where the function is increasing, decreasing, positive, or negative; and maximums and minimums. | complicated cases, to show key features, including: domain and range; rate of change; intercepts; intervals where the function is increasing, decreasing, positive, or negative; maximums and minimums; and end behavior. | quadratic, or exponential) each with a different representation (symbolically, graphically, numerically in tables, or by verbal descriptions). |

## Agenda, Materials, and Preparation

- **Bridge** (*Optional, 5 minutes*)
- **Warm-up** (*5 minutes*)
- **Activity 1** (*15 minutes*)
- **Activity 2** (*10 minutes*)
  - Matching Descriptions to Graphs card sort (print 1 copy per pair of students and cut up in advance)
- **Lesson Debrief** (*5 minutes*)
- **Cool-down** (*5 minutes*)
  - M1.U6.L8 Cool-down (print 1 copy per student)

## LESSON

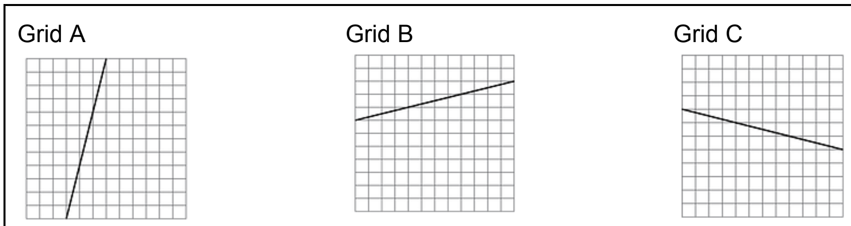
### Bridge (*Optional, 5 minutes*)

**Building On:** NC.8.F.4

To prepare for comparing exponential growth and decay, students will examine linear graphs with positive and negative slopes. First, they will match the graphs with given slopes, and then they will explain how they could recognize a line with a negative slope. Through the discussion, the teacher can sequence responses to ultimately highlight that a specific characteristic of the graph (decreasing from left to right) indicates a negative slope, in the same way that characteristics of exponential graphs indicate growth or decay factors.

#### Student Task Statement

1. Each square on a grid represents 1 unit on each side. Match the numbers with the slopes of the lines.[1]




1. $-\frac{1}{4}$
2. $\frac{1}{4}$
3. $4$

2. Explain how someone could determine the match to grid C just by looking at the graphs and the answer choices.

[1] Adapted from IM 6–8 Math https://curriculum.illustrativemathematics.org/MS/index.html, which was originally developed by Open Up Resources and authored by Illustrative Mathematics, and is copyright 2017–2019 by Open Up Resources. It is licensed under the Creative Commons Attribution 4.0 International License (CC BY 4.0). OUR's 6–8 Math Curriculum is available at https://openupresources.org/math-curriculum/. Adaptations and updates to IM 6–8 Math are copyright 2019 by Illustrative Mathematics, and are licensed under the Creative Commons Attribution 4.0 International License (CC BY 4.0).

| DO THE MATH | PLANNING NOTES |
| --- | --- |
| | |

## Warm-up: Fractions and Decimals (*5 minutes*)

| **Instructional Routine:** Poll the Class | |
| --- | --- |
| **Building On:** NC.6.NS.7 | **Building Towards:** NC.M1.F-IF.9 |

This warm-up provides students with an opportunity to revisit previous experience with fractions to provide greater access to the work in this lesson. Through a medical context, students compare fractions with unlike denominators and may subtract fractions from 1 to find the other part of a whole. When students compare which concentration of medicine is decreasing the fastest, they are building towards comparing exponential functions.

### Step 1

- Read the task with the class. *Poll the Class* with the question, "What is your initial response without doing any calculations?"
- Ask students to arrange themselves in pairs or use visibly random grouping. Encourage them to think quietly for a minute before sharing their responses with their partner.
- After students listen to one another's responses, give them a minute to revise their own response or refine their explanation.

### Student Task Statement

A hospital offers two different medications to treat pain: medication A and medication B.

- The amount of medication A that remains in the body each hour is $\frac{5}{7}$ of the amount in the previous hour.
- The amount of medication B that remains in the body each hour is $\frac{2}{5}$ of the amount in the previous hour.

Which medication leaves the body the fastest? Explain your reasoning.

### Step 2

- Re-poll students asking, "Which students think medication A is decreasing the fastest? Which students think medication B is decreasing the fastest?" Ask students to share their reasoning.

| DO THE MATH | PLANNING NOTES |
|---|---|
| | |

## Activity 1: Falling and Falling (*15 minutes*)

| **Instructional Routine:** Collect and Display (MLR2) | | |
|---|---|---|
| **Building On:** NC.M1.F-IF.4 | **Addressing:** NC.M1.A-CED.2; NC.M1.F-IF.7 | **Building Towards:** NC.M1.F-BF.1a |

Students analyze graphs representing the depreciation of the values of two cell phones. The value of a new cell phone tends to be highest at its initial release and falls over time. Some phones depreciate in a roughly exponential fashion.

Students also write equations to represent the relationship between time and value for each cell phone. To represent the relationships with equations, students need to know the initial value of each phone (which can be read from the graph) and the decay factor (which must be calculated).

Students may conclude that phone B is decreasing in value more quickly than phone A by reasoning in one of the following ways:

- Phone A loses $\frac{2}{5}$ of its value each year while phone B loses $\frac{1}{2}$ of its value, so the value of phone B is decreasing more quickly.
- Visually, the value of phone B has a more drastic dropoff than the value of phone A. With each consecutive year, the points on phone B's graph get twice as close to the horizontal axis, while consecutive points on phone A's graph do not approach the horizontal axis as quickly.

Students who compare differences instead of factors may conclude that the value of phone A is decreasing faster. (The differences for phone A are 400, 240, and 144, while the differences for phone B are 400, 200, and 100.)

Some thoughts that may be helpful for the discussion in Step 2 (or as needed):

- For depreciation, we're interested in how items lose their value as a percentage of the original price, so comparing using factors is appropriate.
- Also, if phone A really is losing value faster than phone B, then at some point in the future, phone A should cost less than phone B. Calculating the value of each phone over the next few years should convince students that this will never happen.

### Step 1

- To give students an overview of the context, consider sharing a news clip or advertisement on the latest release of a popular cell phone or ask students to share what they know about the latest models of some phones.
  - Solicit ideas from students about how they think the value of a phone changes after it is released.
  - Address any unfamiliar words or phrases during this discussion, such as "initial release," and remind students what "depreciation" means, if needed.
- Students are likely to be familiar with the idea that phones decrease in value over time (especially as newer ones come along). Give an example of how a popular phone might have cost, for instance, $1000 when it was first available to the public, but the same type of phone (in new condition) might cost several hundred dollars less a couple of years later.

## Step 2

- Have students continue to work with their partner from the warm-up.
  - Students should work individually on problems 1 and 2, then compare their work and collaboratively agree on a response for problem 3.
  - Students should make sure they agree on the meaning of “depreciate by the same factor” before working together on problems 4 and 5.

Using the *Collect and Display* routine, listen for and collect vocabulary and phrases students use to describe depreciation.

- Write the students’ words on a visual display and update it throughout the remainder of the lesson.
- Be sure to emphasize the term “decay factor” along with associated words used to calculate it, such as “consecutive values” and “dividing.”
- Remind students to borrow language from the display as needed. This will help students read and use mathematical language during their partner and whole-group discussions.

**RESPONSIVE STRATEGY**

Display or build an anchor chart with an exponential function and the meaning of each variable. To increase access to the chart, each variable can be color coded. Students can be encouraged to utilize the chart by highlighting key values and parts of their own equations and graphs with the same colors displayed on the chart.

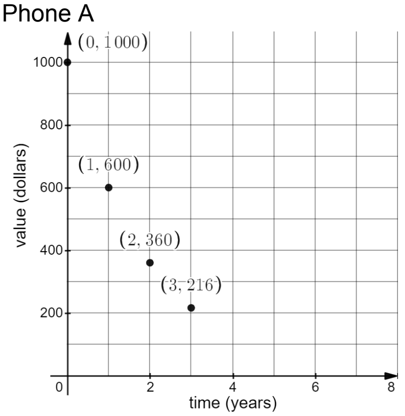
Supports accessibility for:
Conceptual processing; Memory

***Advancing Student Thinking***: Students may not see that the relationship is exponential. Ask them by what factor the value of each phone decreases in the first year. What about in the second year? Ask them what these numbers tell them about the decay factor.

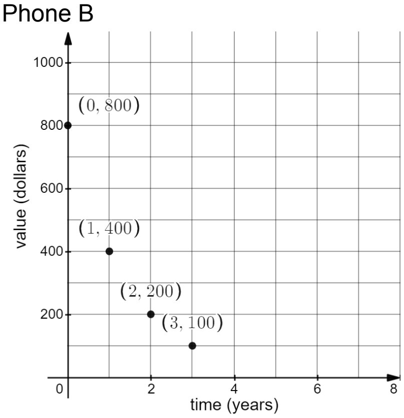
### Student Task Statement

The value of some cell phones changes exponentially after initial release. Here are graphs showing the depreciation of two phones 1, 2, and 3 years after they were released.

Phone A




Phone B




1. Which phone is more expensive to buy when it is first released?
2. How does the value of each phone change with every passing year?
3. Which one is decreasing in value more quickly? Explain or show how you know.
4. If the phones continue to depreciate by the same factor each year, what will the value of each phone be 4 years after its initial release?
5. For each cell phone, write an equation that relates the value of the phone in dollars to the years since release, $t$. Use $v$ for the value of phone A and $w$ for the value of phone B.

**Are You Ready For More?**

When given data, it is not always clear how to best model it. In this case, we were told the value of the cell phones was changing exponentially. Suppose, however, we were instead given only the initial values of the cell phones when released and the values after each of the first three years.

1. Use technology to compute the best fit line for each cell phone. Round any numbers to the nearest dollar.
2. Explain why, in this situation, an exponential model might be more appropriate than the linear model you just created.

### Step 2

- Invite students to share their responses. Focus the conversation on how they determined which phone loses value more quickly and on how they wrote the equations, referring back to—and adding to—the student language collected and displayed, where relevant.
- Make sure students recognize that, to tell which phone depreciates more quickly, they need to identify a "decay factor" and cannot simply find the differences between consecutive values. Instead of subtracting, they should be dividing the value of one year by that of the previous year. Once the initial amount and the decay factor are identified, they can write an equation to predict the value of the phone later in time (assuming the mathematical model still applies, e.g., the technology is not considered obsolete or the phone doesn't somehow become more valuable because of other extraneous reasons).

| DO THE MATH | PLANNING NOTES |
| --- | --- |
| | |

## Activity 2: Matching Descriptions to Graphs (*10 minutes*)

| **Instructional Routines:** Card Sort; Take Turns; Discussion Supports (MLR8) - Responsive Strategy | | |
| --- | --- | --- |
| **Building On:** NC.M1.F-IF.4 | **Addressing:** NC.M1.F-IF.7 | **Building Towards:** NC.M1.F-IF.9 |

In this *Card Sort* activity, students match four descriptions of situations characterized by exponential change with four graphs. In order to make a correct match, students will need to attend to whether the multiplier is greater than 1 or between 0 and 1, and determine the growth or decay factor from the graph with enough precision to distinguish between two possibilities.

Refer to the blackline master for planning for the card sort.

**RESPONSIVE STRATEGY**

Provide cards with key elements on the descriptions and graphs pre-highlighted to assist students in matching.

Supports accessibility for:
Memory; Visual-spatial processing

| CARD SORT | ***What Is This Routine?*** A *Card Sort* uses cards or slips of paper that can be manipulated and moved around (or the same functionality enacted with a computer interface). It can be done individually or in small groups. Students put things into categories or groups based on shared characteristics or connections. This routine can be combined with *Take Turns*, such that each time a student sorts a card into a category or makes a match, they are expected to explain the rationale while the group listens for understanding. The first few times students engage in these activities, the teacher should demonstrate how the activity is expected to go. Once students are familiar with these structures, less set-up will be necessary. While students are working, the teacher can ask students to restate their question more clearly or paraphrase what their partner said.<br><br>***Why This Routine?*** A *Card Sort* provides opportunities to attend to mathematical connections using representations that are already created, instead of expending time and effort generating representations. It gives students opportunities to analyze representations, statements, and structures closely and make connections (MP2, MP7). |
|---|---|

### Step 1

- Ask students to arrange themselves pairs or use visibly random grouping. Give each group a set of slips from the card sort.
- Ask students to *Take Turns*. The first partner identifies a match and explains why they think it is a match, while the other listens and works to understand. Then they switch roles. Before students begin working, model this routine if necessary.

**RESPONSIVE STRATEGY**

While students take turns finding a match and explaining their reasoning to their partner, display the following sentence frames for all to see: “ ____ and ____ represent the same situation because....”, and “I noticed ____ , so I matched....” Encourage students to challenge each other when they disagree. This will help students clarify their reasoning about using the factor or multiplier to match an exponential situation represented by a description or graph.

Discussion Supports (MLR8)

**Monitoring Tip:** Here are some strategies students may use to match. Look for students using these or other approaches and let them know that they may be asked to share later. Include at least one student who does not typically volunteer.

- Separate the cards and descriptions into situations that grow or decay, and then analyze the graphs more closely to compare their growth factors.
- Use the information in the descriptions to draw conclusions about the coordinates of points on the graphs.

***Advancing Student Thinking***: If students struggle to identify the correct graphs, ask them to start by grouping the cards/situations that involve growth, and the cards/situations that involve decay. To narrow it down further, call students’ attention to phrases like “doubles every 4 years” and “loses $\frac{1}{4}$ of its value each year,” and encourage them to find points within each graph that they can compare to see if these phrases apply.

### Student Task Statement

Your teacher will give you a set of cards containing descriptions of situations and graphs. Match each situation with a graph that represents it. Record your matches and be prepared to explain your reasoning.

### Step 2

- Select students to share their strategies, starting with the cards that suggest a growth factor (card 1 and card 5) and then moving to cards that suggest a decay factor (card 2 and card 6). If not mentioned by students, discuss questions such as:

**RESPONSIVE STRATEGY**

Provide students with a two-column graphic organizer to record deeper understandings about the features of exponential graphs and the related findings revealed in the card sort discussion.

Supports accessibility for: Language; Organization

– "Can we use the vertical intercepts to make a match?" (No, that information is not given in the descriptions of the situations. Plus, some of the graphs have the same vertical intercept.)

– "What can we tell about the growth or decay factor in each situation?" (The growth happens more quickly for the situation in card 1 than in card 5 because the stock described in card 1 quadruples every eight years. For cards 2 and 6, the phone loses $\frac{2}{5}$ of its value every year, while the car only loses $\frac{1}{4}$ of its value each year.)

**DO THE MATH**

**PLANNING NOTES**

## Lesson Debrief (*5 minutes*)

The purpose of this lesson is to review how we can gather key information about situations from graphs representing them, with a focus on exponential decay.

Choose whether students should first have an opportunity to reflect on the following questions in their workbooks or talk through them with a partner. Display one graph from the lesson, and ask:

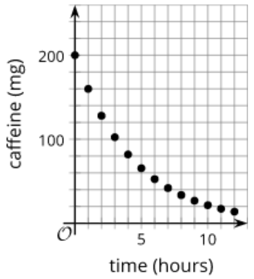
- "How can you use the graph to determine the initial amount?" (The $y$-value of the vertical intercept of the graph provides the initial amount.)
- "How can you tell whether the graph shows exponential growth or exponential decay?" (The graph increases if there is exponential growth, but decreases if there is exponential decay.)
- "How can you use the graph to determine the growth or decay factor?" (You can determine the growth or decay factor from a graph by finding the quotient of one $y$-value and the previous $y$-value.)

**PLANNING NOTES**

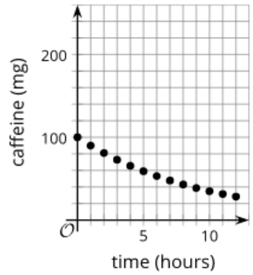
### Student Lesson Summary and Glossary

Graphs are useful for comparing relationships. Here are two graphs representing the amount of caffeine in the bloodstreams of Person A and Person B, in milligrams, at different times, measured hourly, after an initial measurement.

Person A




Person B



The graphs reveal interesting information about the amount of caffeine in each person over time:

- At the initial measurement, Person A's bloodstream has more caffeine ($200$ milligrams) than Person B's ($100$ milligrams).
- The caffeine in Person A's bloodstream decreases faster. It went from $200$ to $160$ milligrams in an hour. Because $160$ is $\frac{8}{10}$ or $\frac{4}{5}$ of $200$, the decay factor is $\frac{4}{5}$.
- The caffeine in Person B's bloodstream went from $100$ to about $90$ milligrams in an hour, so that decay factor is about $\frac{9}{10}$. This means that after each hour, a larger fraction of caffeine stays in Person B's bloodstream than in Person A's.
- Even though Person A started out with twice as much caffeine in their bloodstream, because of the decay factor, Person A's bloodstream had less caffeine than Person B's after 6 hours.

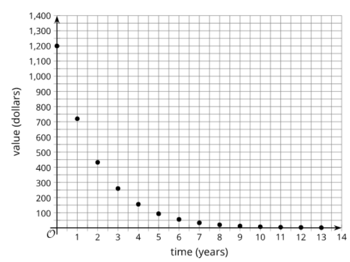
## Cool-down: A Phone, a Company, a Camera (*5 minutes*)

**Addressing:** NC.M1.F-IF.7

**Cool-down Guidance:** Points to Emphasize
If a significant number of students choose option a, consider doing a think-aloud in class to determine the rate of decay. Assign a practice problem from this lesson for students to consolidate their thinking. If students struggle to explain their reasoning, select a few examples of student work and discuss as a class what makes the reasoning clear or how the reasoning might be strengthened.

### Cool-down

1. This graph represents one of the following descriptions. Which one?




   a. A phone loses $\frac{4}{5}$ of its value every year after purchase: the relationship between the number of years since purchasing the phone and the value of the phone.
   b. The number of stores a company has tripled approximately every 5 years: the relationship between the number of years and the number of stores.
   c. A camera loses $\frac{2}{5}$ of its value every year after purchase: the relationship between the number of years since purchasing the camera and the value of the camera.
2. Explain how you know the graph represents the description you chose.

**Student Reflection:**

What personal strengths helped you to work through the lesson today? What do you wish you could do better?

**DO THE MATH**

**INDIVIDUAL STUDENT DATA**

**SUMMARY DATA**

**NEXT STEPS**

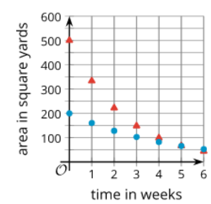
## TEACHER REFLECTION

**What went well in the lesson? What would you do differently next time? What happened today that will influence the planning of future lessons?**

**What part of the lesson went really well today in terms of students' learning? What did you do that made that part go well?**

## Practice Problems

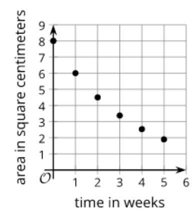
1. The two graphs show models characterized by exponential decay representing the area covered by two different algae blooms, in square yards, $w$ weeks after different chemicals were applied.




   a. Which algae bloom covered a larger area when the chemicals were applied? Explain how you know.

   b. Which algae population is decreasing more rapidly? Explain how you know.

2. A medicine is applied to a burn on a patient's arm. The area of the burn in square centimeters decreases exponentially and is shown in the graph.




   a. What fraction of the burn area remains each week?

   b. Write an equation representing the area of the burn, $a$, after $t$ weeks.

   c. What is the area of the burn after 7 weeks? Round to three decimal places.

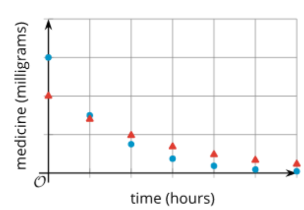
3.

   a. The area of a sheet of paper is 100 square inches. Write an equation that gives the area, $A$, of the sheet of paper, in square inches, after being folded in half $n$ times.

   b. The area of another sheet of paper is 200 square inches. Write an equation that gives the area, $B$, of this sheet of paper, in square inches, after being folded into thirds $n$ times.

   c. Are the areas of the two sheets of paper ever the same after each being folded $n$ times? Explain how you know.

4. The graphs show the amounts of medicine in two patients after receiving injections. The circles show the medicine in Patient A and the triangles show that in Patient B.




   One equation that gives the amount of medicine in milligrams, $m$, in Patient A, $h$ hours after an injection, is $m = 300(\frac{1}{2})^h$.

   What could be an equation for the amount of medicine in Patient B?

   a. $m = 500(\frac{3}{10})^h$

   b. $m = 500(\frac{7}{10})^h$

   c. $m = 200(\frac{3}{10})^h$

   d. $m = 200(\frac{7}{10})^h$

5. Select **all** expressions that are equivalent to $3^8$.

   a. $3^2 \cdot 3^4$
   b. $3^2 \cdot 3^6$
   c. $\frac{3^{16}}{3^2}$
   d. $\frac{3^{12}}{3^4}$
   e. $(3^4)^2$
   f. $(3^1)^7$

   (From Unit 6, Lessons 1 and 2)

6. Priya simplifies $(\frac{x}{x^{-4}})^3$ to $x^{15}$ using the following steps:

   - Step 1: $(\frac{x^3}{x^{-12}})$
   - Step 2: $(x^3 \cdot x^{12})$
   - Step 3: $x^{15}$

   Han simplifies the same expression to $x^{15}$, and he uses a different series of steps. What steps might Han have used?

   (From Unit 6, Lesson 2)

7. (*Technology required.*) Use a graphing calculator to determine the equation of the line of best fit. Round numbers to two decimal places.

| $x$ | 10 | 12 | 15 | 16 | 18 | 20 | 24 |
|---|---|---|---|---|---|---|---|
| $y$ | 27 | 22 | 21 | 19 | 15 | 14 | 10 |

   (From Unit 4)

8. The data set represents the number of students in several different classes who scored a perfect score on the most recent math test.

   12, 14, 15, 15, 17, 19, 20, 30

   a. What is the mean of the data set? Interpret this value in the situation.

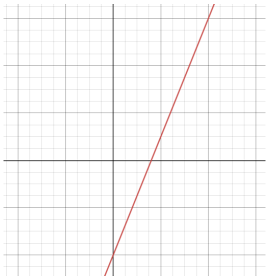
   b. What is the median of the data set? Interpret this value in the situation.

   c. Is there an outlier? How does it impact the mean compared to the median?

   (From Unit 1)

9. What is the slope of the line below? How do you know?

   (Addressing NC.8.F.4)



# Lessons 9 & 10: Checkpoint

## PREPARATION

| Lesson Goals | Learning Targets |
|---|---|
| • Learn and grow mathematically in course-level content.<br>• Communicate and address mathematical areas of strength and areas of growth. | • I can continue to grow as a mathematician and challenge myself.<br>• I can share what I know mathematically. |

### Lesson Narrative

This is a Checkpoint day. Checkpoint days consist of two lessons (one full block) and are structured as four 20-minute stations that students rotate between. There are a total of six stations students can engage with. Since students will not be able to participate in all six stations, please note that Station A (Unit 7 Check Your Readiness) is required for all students.

This Checkpoint does not include an Are You Ready For More? Station, as many of these additional opportunities are not stand-alone activities but connect back to the activities in the given lesson. If students did not previously complete the Are You Ready For More? activities, an additional station option would be for students to do so today.

A. Unit 7 Check Your Readiness (*Required*)
B. Teacher-led Small-group Instruction
C. Recalling Percent Change
D. The Legacy of Henrietta Lacks
E. I Know My Exponents!
F. Micro-Modeling with Pizza
G. Studying Bacterial Growth

**How will you determine the topics for small-group instruction?**

**How will you create a space for students to share and reflect on what they learned about Henrietta Lacks?**

## Agenda, Materials, and Preparation

- **Station A** (*Required, 20 minutes*)
  - Unit 7 Check Your Readiness (print 1 copy per student)
- **Station B** (*20 minutes*)
- **Station C** (*20 minutes*)
- **Station D** (*20 minutes*)
  - “Henrietta Lacks: Her Impact and Our Outreach” video: https://bit.ly/HonoringHenrietta
  - “Henrietta Lacks: science must right a historical wrong” editorial: https://go.nature.com/3k0vhlm
- **Station E** (*20 minutes*)
  - Optional Desmos version of activity: https://bit.ly/M1U6L9exponents
- **Station F** (*20 minutes*)
- **Station G** (*20 minutes*)

## STATIONS

### Station A: Unit 7 Check Your Readiness (*Required, 20 minutes*)

Remind students that it is really important that their responses to these questions accurately represent what they know. Ask them to answer what they can to the best of their ability. If they get stuck, they should name what they don’t know or understand.

### Station B: Teacher-led Small-group Instruction (*20 minutes*)

Use student cool-down data, Check Your Readiness Unit 6 data, and informal formative assessment data from Unit 6 (Lessons 1–8) to provide targeted small-group instruction to students who demonstrate the need for further support or challenge on topics taught up to this point.

Potential topics:

- Exponent properties
- Recalling percent change (using materials from Station C)
- Representing exponential growth and decay functions
- Analyzing exponential growth and decay graphs

### Station C: Recalling Percent Change (*20 minutes*)

| | |
|---|---|
| **Building On:** NC.7.EE.3 | **Building Towards:** NC.M1.F-BF.1a |

The goal of this station is to remind students about what they know of percent change and the different ways of writing expressions for it (a topic from grade 7), in preparation for the situations they will encounter in upcoming lessons. A repeated percent increase or decrease is an exponential change. This station aligns to question 7 in Check Your Readiness.

#### Student Task Statement

1. You need to pay 8% tax on a car that costs $12,000. What will you end up paying in total? Show your reasoning.
2. Burritos are on sale for 30% off. Your favorite burrito normally costs $8.50. How much does it cost now? Show your reasoning.
3. A pair of shoes that originally costs $79 is on sale for 35% off. Does the expression $0.65(79)$ represent the sale price of the shoes (in dollars)? Explain your reasoning.
4. Come up with some strategies for mentally adding 15% to the total cost of an item.

5. Complete the table so that each row has a description and two different expressions that answer the question asked in the description. The second expression should use only multiplication. Be prepared to explain how the two expressions are equivalent.

| Description and question | Expression 1 | Expression 2 (using only multiplication) |
|---|---|---|
| A one-night stay at a hotel in Anaheim, CA costs $160. Hotel room occupancy tax is 15%. What is the total cost of a one-night stay? | $160 + (0.15) \cdot 160$ | |
| Teachers receive a 30% discount at a museum. An adult ticket costs $24. How much would a teacher pay for admission into the museum? | | $(0.7) \cdot 24$ |
| The population of a city was 842,000 ten years ago. The city now has 2% more people than it had then. What is the population of the city now? | | |
| After a major hurricane, 46% of the 90,500 households on an island lost their access to electricity. How many households still have electricity? | | |
| | $754 - (0.21) \cdot 754$ | |
| Two years ago, the number of students in a school was 150. Last year, the student population increased 8%. This year, it increased by 8% again. What is the number of students this year? | | |

## DO THE MATH

## PLANNING NOTES

## Station D: The Legacy of Henrietta Lacks(*20 minutes*)

In Unit 6, Lesson 5, Activity 1, students were introduced to HeLa cells without the full context. HeLa cells have been, and continue to be, invaluable for medical research since 1951. For decades after her death in 1951, the family of Henrietta Lacks, was not informed about the significant use of HeLa cells, despite the fact that her name and medical records were shared with media and throughout the medical community. Even though the medical discoveries made possible by these cells contribute to continued profits for companies that produce medications and treatments, the family of Henrietta Lacks has not been compensated. This activity provides students with an opportunity to consider the magnitude of even a small offering of compensation, and to reflect on the potential impact of compensation for Lacks' descendants. Teachers doing Station D should consider if they want to make the station a whole-class activity, reserve class time to process the activity, or enlist another adult to monitor the station.

Students are asked to take 10 minutes to learn more about Henrietta Lacks and HeLa cells. A few options to learn more include:

- "Henrietta Lacks: Her Impact and Our Outreach" video: https://bit.ly/HonoringHenrietta
- "Henrietta Lacks: science must right a historical wrong" editorial: https://go.nature.com/3k0vhlm

### Station D

In Unit 6, Lesson 5, Activity 1, you were introduced to HeLa cells, with a unique characteristic of doubling about every 24 hours, used for medical research. Take 10 minutes to learn more about Henrietta Lacks and HeLa cells, and then answer the following questions:

1. Suppose the family of Henrietta Lacks was given $1,000 as compensation for using HeLa cells for research once it was discovered how valuable they were in 1951. Use the question and prompts below to determine the amount this compensation would be worth to the family today if it accrued interest at a rate of 10% per year.
    a. If an amount of money grows 10% per year, what is the annual growth factor?
    b. Create an exponential equation for a function representing the value of compensation as a function of the number of years since 1951, with an initial value of $1,000, and the growth factor from part a.
    c. Calculate the value of $1,000 today, using the number of years since 1951.
2. Do you believe the family of Henrietta Lacks deserves compensation for the contributions she unknowingly made to modern medicine? If so, how much do you think is owed to the family? Share your reasoning.
3. Do you think requiring consent for research should be required, meaning an individual or family has the right to agree or refuse permission to use samples taken during treatment? Describe why you believe consent is a good or bad requirement for medical research.

| DO THE MATH | PLANNING NOTES |
| --- | --- |
| | |

## Station E: I Know My Exponents! (*20 minutes*)

Students will use integers to make sense of equations using exponents. They will construct their own equations using a set of integers to make them true. This activity can be done using technology (Desmos) or by hand in their Student Workbook.

### Step 1

- If students are using the Desmos version of this activity, prior to class, go to https://bit.ly/M1U6L9exponents to access the Activity Builder for this Desmos station.
- Give students access to this station by clicking on "Assign" and choose either "Assign to Your Classes" or "Single Session Code."
  - A class must be created, and students added to it, in a teacher's Desmos account in order to use the "Assign to Your Classes" option. Students will see a "Start" button next to the activity title when logged in on the student.desmos.com page.
  - In order to do this activity without creating a class in Desmos, a "Single Session Code" can be generated to give to students. Instruct students to go to student.desmos.com and enter the single session code.

### Step 2

- Student progress can be monitored by clicking "View Dashboard" underneath Activity Sessions on the Desmos Activity Builder page. From this dashboard, student pacing can be adjusted, the activity can be paused for students, and student names can be anonymous.
- Provide feedback to individual students by clicking the chat icon at the top of the student work window of a particular slide.

### Station E

1. In these puzzles, you will fill in the boxes to make true equations involving exponents. The example on the right shows a completed puzzle. Using only numbers 0–9, can you find two other combinations that work?

$$\boxed{3}^{\boxed{2}} = \boxed{9}$$

a. $\square^{\square} = \square$

b. $\square^{\square} = \square$

2. Use any values between 0–9 to make the equations true.

a. $3^{\square} \cdot 3^{\square} = 3^{\square}$

b. $3^{\square} \cdot 3^{\square} = 9^{\square}$

c. $3^{\square} \cdot 3^{\square} = \square^{\square}$

d. $(x^{\square})^{\square} = x^{\square}$

e. $(\square x^{\square})^{\square} = x^{\square}$

f. $\square^{\square} \cdot \square^{\square} = \square^{\square}$

g. $x^{\square} y^{\square} \cdot xy^{\square} = x^{\square} y^{\square}$

3. Use any values between $-4$ and $9$ to make the equations true.

a. $\dfrac{3^{\square}\cdot 3^{\square}}{3^{\square}} = 3^{\square}$

b. $(\square\cdot 10^{\square})\,(\square\cdot 10^{\square}) = \square\cdot 10^{\square}$

c. $\dfrac{(\square\cdot 10^{\square})}{(\square\cdot 10^{\square})} = \square\cdot 10^{\square}$

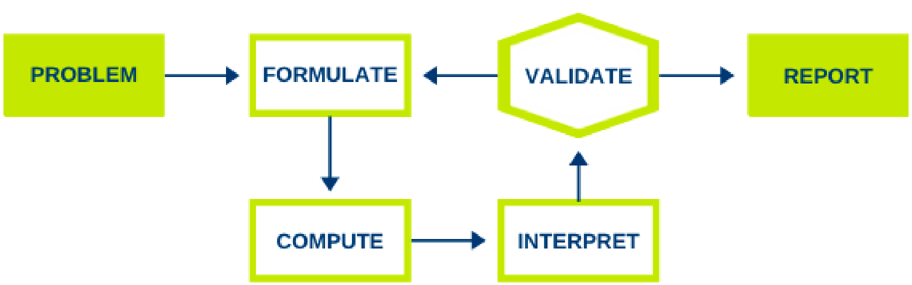
**DO THE MATH**

**PLANNING NOTES**

## Station F: Micro-Modeling with Pizza (*20 minutes*)

**Instructional Routine:** Aspects of Mathematical Modeling

Modeling is the link between the mathematics students learn in school and the problems they will face in college, career, and life. Time spent on modeling in Math 1 is crucial, as it prepares students to use math to handle technical subjects in their further studies, and problem solve and make decisions that adults regularly encounter in their lives.




### Station F

A pizzeria on Planet Z serves two round pizzas of the same thickness in different sizes. The smaller one has a diameter of 30 cm and costs 30 zeds. The larger one has a diameter of 40 cm and costs 40 zeds. Which pizza is a better value for money? Show your reasoning.[1]

[1] Adapted from https://tasks.illustrativemathematics.org/

**DO THE MATH**

**PLANNING NOTES**

## Station G: Studying Bacterial Growth (*20 minutes*)

**Building On:** NC.8.EE.1

Station G provides students with an opportunity to engage with a task that leads students through the use of repeated reasoning to understand algorithms and make generalizations about patterns (MP8).

### Station G

Mai and Tyler are lab partners studying bacterial growth. They were surprised to find that the population of the bacteria doubled every hour.[2]

1. The table shows that there were 2,000 bacteria at the beginning of the experiment. What was the size of the population of bacteria after 1 hour? After 2, 3, and 4 hours? Enter this information into the table:

| **Hours into study** | | | | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|
| **Population (thousands)** | | | | 2 | | | | |

2. If you know the size of the population at a certain time, how do you find the population one hour later?
3. Mai said she thought that they could use the equation $P = 2t + 2$ to find the population at time $t$. Tyler said they thought that they could use the equation $P = 2 \cdot 2^t$. Decide whether either of these equations produces the correct populations for $t = 1, 2, 3, 4$.
4. Assuming the population doubled every hour before the study began, what was the population of the bacteria 1 hour *before* the students started their study? What about 3 hours before?
5. If you know the size of the population at a certain time, how do you find the population 1 hour *earlier*?
6. What number would you use to represent the time 1 hour before the study started? 2 hours before? 3 hours before? Finish filling in the table if you haven't already.
7. Now use Tyler's equation to find the population of the bacteria 1 hour before the study started. Use the equation to find the population of the bacteria 3 hours before. Do these values produce results consistent with the arithmetic you did earlier?
8. Use the context to explain why it makes sense that $2^{-n} = (\frac{1}{2})^n = \frac{1}{2^n}$. That is, describe why, based on the population growth, it makes sense to define 2 raised to a negative integer exponent as repeated multiplication by $\frac{1}{2}$.

[2] Adapted from https://tasks.illustrativemathematics.org/

## DO THE MATH

## PLANNING NOTES

## TEACHER REFLECTION

**What went well in the lesson? What would you do differently next time? What happened today that will influence the planning of future lessons?**

**When do your students feel successful in math? How do you know?**

# Lesson 11: Exponential Situations as Functions

## PREPARATION

| Lesson Goals | Learning Targets |
| --- | --- |
| • Determine and explain (orally and in writing) whether relationships—in descriptions, tables, equations, or graphs—are functions.<br>• Use function notation to write equations that represent exponential relationships. | • When I see relationships in descriptions, tables, equations, or graphs, I can determine whether the relationships are functions.<br>• I can use function notation to write equations that represent exponential relationships. |

### Lesson Narrative

Prior to this lesson, students studied situations characterized by exponential change using descriptions, tables, graphs, and equations. In this lesson, they start to view these relationships as **exponential functions**. This means choosing an independent and dependent variable and expressing the relationships using function language and, in some cases, function notation. For an exponential relationship, either variable can be the independent variable, but one choice gives an exponential function while the other gives a logarithmic function (which is outside the scope of this course). The contexts here are chosen and presented so that it is more natural to choose an independent variable that leads to an exponential function.

Students use variables to represent one real-world quantity as a function of another, which is an example of decontextualizing and reasoning abstractly (MP2).

**What math language will you want to support your students with in this lesson? How will you do that?**

### Focus and Coherence

| Building On | Addressing |
| --- | --- |
| **NC.7.G.4:** Understand area and circumference of a circle.<br>• Understand the relationships between the radius, diameter, circumference, and area.<br>• Apply the formulas for area and circumference of a circle to solve problems.<br><br>(*continued*) | **NC.M1.F-IF.2:** Use function notation to evaluate linear, quadratic, and exponential functions for inputs in their domains, and interpret statements that use function notation in terms of a context. |

| | |
|---|---|
| **NC.8.G.9:** Understand how the formulas for the volumes of cones, cylinders, and spheres are related and use the relationship to solve real-world and mathematical problems.<br><br>**NC.M1.F-IF.1:** Build an understanding that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range by recognizing that:<br>• if $f$ is a function and $x$ is an element of its domain, then $f(x)$ denotes the output of $f$ corresponding to the input $x$.<br>• the graph of $f$ is the graph of the equation $y = f(x)$.<br><br>**NC.M1.A-CED.2:** Create and graph equations in two variables to represent linear, exponential, and quadratic relationships between quantities. | **NC.M1.F-IF.5:** Interpret a function in terms of the context by relating its domain and range to its graph and, where applicable, to the quantitative relationship it describes.<br><br>**NC.M1.F-IF.7:** Analyze linear, exponential, and quadratic functions by generating different representations, by hand in simple cases and using technology for more complicated cases, to show key features, including: domain and range; rate of change; intercepts; intervals where the function is increasing, decreasing, positive, or negative; maximums and minimums; and end behavior.<br><br>**NC.M1.F-BF.1a:** Build linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two ordered pairs (include reading these from a table). |

## Agenda, Materials, and Preparation

- **Bridge** (*Optional, 5 minutes*)
- **Warm-up** (*5 minutes*)
- **Activity 1** (*15 minutes*)
  - Technology is required for this activity: Acquire devices that can access Desmos (recommended) or other graphing technology. It is ideal if each student has their own device.
- **Activity 2** (*10 minutes*)
- **Lesson Debrief** (*5 minutes*)
- **Cool-down** (*5 minutes*)
  - M1.U6.L11 Cool-down (print 1 copy per student)

## LESSON

### Bridge (*Optional, 5 minutes*)

**Building On:** NC.7.G.4; NC.8.G.9

In this bridge, students will have the opportunity to review the concept of a function. Rather than starting with exponential functions, students examine the familiar formulas for area of a circle and volume of a sphere. As students share their answers, discuss how there is only one unique area or volume (output) for each radius (input), ensuring that the formula represents a function.

#### Student Task Statement

1. The area, $A$, of circle $C$ is given as a function of the radius, $r$, based on the rule $A = \pi r^2$. What is the area of circle $C$ given radius:

   a. $5$ inches? b. $12$ inches?

2. The volume, $V$, of sphere $S$ is given as a function of the radius, $r$, based on the rule $V = \frac{4}{3}\pi r^3$. What is the volume of sphere $S$ given radius:

   a. $5$ inches? b. $12$ inches?

| DO THE MATH | PLANNING NOTES |
|---|---|
| | |

## Warm-up: Rainfall in Las Vegas (*5 minutes*)

| |
|---|
| **Instructional Routine:** Notice and Wonder |
| **Building On:** NC.M1.F-IF.1 |

The goal of this warm-up is to review the meaning of a function presented graphically. While students do not need to use function notation here, interpreting the graph in terms of the context will prepare them for their work with functions in the rest of the unit.

### Step 1

- Ask students to close their books or devices. Display the graph for all to see.

  Ask students to observe the graph and be prepared to share things they *Notice and Wonder*. Select students to briefly share one thing they noticed or wondered to help ensure all students understand the information conveyed in the graph.

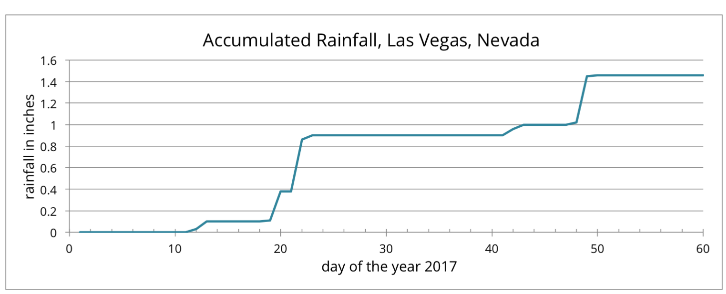
- Ask students to open their books or devices and answer the questions about the graph. Follow with a whole-class discussion.

***Advancing Student Thinking***: If students struggle to see from the graph how the accumulated rainfall is a function of time but time is not a function of accumulated rainfall, consider displaying the data in a table. Shown here are the data for the first 20 days of 2017. Help students see that for every value of $t$, the time in days, there is one value of $r$, the accumulated rainfall in inches, but this is not true the other way around.

| $t$ (days) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $r$ (inches) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0.03 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.1 | 0.11 | 0.38 |

### Student Task Statement

Here is a graph of the accumulated rainfall in Las Vegas, Nevada, in the first 60 days of 2017.




Use the graph to support your answers to the following questions.

1. Is the accumulated amount of rainfall a function of time?
2. Is time a function of accumulated rainfall?

### Step 2

- Make sure students understand why the accumulated rain is a function of time but not the other way around.
- Recall the notation for writing, using function notation, accumulated rainfall as a function of time. If $r$ represents the amount of rainfall in inches and $t$ is time in days, then $r(t)$ is the amount of rain that has fallen in the first $t$ days of 2017. For example, $r(2) = 0$ tells us that the accumulated rainfall in the first two days of the year was 0 inches. $r(48) = 1$ means that there was 1 inch of accumulated rain in the first 48 days of the year.
- Ask students to write and explain the meaning of a few other statements using function notation. For example, $r(30) = 0.9$ and $r(40) = 0.9$, meaning that between days 30 and 40, it did not rain and the accumulated rainfall stayed at 0.9 inches.

| DO THE MATH | PLANNING NOTES |
|---|---|
| | |

## Activity 1: Moldy Bread (*15 minutes*)

| **Instructional Routines:** Graph It; Collect and Display (MLR2) | |
|---|---|
| **Building On:** NC.M1.F-IF.1; NC.M1.A-CED.2 | **Addressing:** NC.M1.F-IF.5; NC.M1.F-IF.7; NC.M1.F-BF.1a |

In this *Graph It* activity, students represent a situation using a table of values, a graph, and an equation. From the exponential equation, it is a short step to thinking of the relationship between the quantities as a function.

Note that it is possible and acceptable to think of time as a function of area, but expressing this using an equation is out of the scope of this course. Students could, however, represent such a function with a graph, table, or description.

Making graphing technology available gives students an opportunity to choose appropriate tools strategically (MP5).

### Step 1

- Ask for a volunteer to read the context for this activity, then ask for another volunteer to summarize in their own words.
- Give students 3 minutes to work on this activity individually, then work in pairs to complete the activity, starting by sharing their individual thoughts.
- During the partner discussions, use the *Collect and Display* routine to listen for and collect the language students use to describe the situation as a function. Call students' attention to language such as "independent or dependent variable" or "input or output value." Write the students' words and phrases on a visual display and update it throughout the remainder of the lesson.

**RESPONSIVE STRATEGY**

Represent the same information through different modalities by drawing a diagram. Encourage students who are unsure where to begin to sketch a diagram of a slice of bread on graph paper and to shade the area that is covered in mold after 1 day, then 2 days . . . until they reach the day when the slice of bread is completely covered in mold.

Supports accessibility for: Conceptual processing; Visual-spatial processing

- Remind students to borrow language from the display as needed. This will help students use mathematical language for describing an exponential function and determining which variable is a function of the other.

***Advancing Student Thinking***: Students may have trouble understanding how to account for time in the first question. They may benefit from writing the area after 1 day has passed, 2 days have passed, etc. A table is a convenient way to gather this information.

## Student Task Statement

Clare noticed mold on the last slice of bread in a plastic bag. The area covered by the mold was about 1 square millimeter. She left the bread alone to see how the mold would grow. The next day, the area covered by the mold had doubled, and it doubled again the day after that.

1. If the doubling pattern continues, how many square millimeters will the mold cover 4 days after she noticed the mold? Show your reasoning.
2. Represent the relationship between the area $A$, in square millimeters, covered by the mold and the number of days $d$ since the mold was spotted using:
   a. a table of values, showing the values from the day the mold was spotted through 5 days later
   b. an equation
   c. a graph

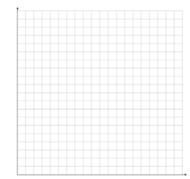
a.

| $d$ | $A$ |
|---|---|
| | |
| | |
| | |
| | |
| | |
| | |

b.

Equation: ___________

c.



3. Discuss with your partner: Is the relationship between the area covered by mold and the number of days a function? If so, write "________ is a function of ________." If not, explain why it is not.

## Are You Ready For More?

What do you think is an appropriate domain for the mold area function $A$? Explain your reasoning.

### Step 2

- Facilitate a whole-class discussion on why the area covered by mold is a function of the number of days that have passed. Refer to any useful language collected and displayed, and, where relevant, connect student words and phrases with the language students learned in the prior unit on functions. For example: The area of the mold $A$ is a function of the number of days $d$ since the mold was spotted, $A = f(d)$. The function $f$ expressing the mold relationship can be written as $f(d) = 1 \cdot 2^d$, where $d$ measures days since the mold was spotted and $f(d)$ gives the area covered by the mold in square millimeters.
- Discuss whether a discrete graph or a curve is more appropriate and what domain would be suitable in this context. Note that up to this lesson, all graphs have been discrete, though students may have graphed them continuously with technology. Ask questions such as:
  - "Can the independent variable be something like 1.5, a number that is not a whole number? Is there an area that is associated with 1.5 days?" (Yes, some area of the bread is covered by mold at any point in time. The mold doesn't disappear after being spotted and then reappear at exactly 1 full day, 2 full days, etc.)
  - "What would be the meaning of a point on the graph where the value of $d$ is, for instance, between 2 and 3?" (It would mean the area covered by mold at some point longer than 2 days but fewer than 3 days after mold was spotted.)
  - "What domain would be appropriate for this function? Can the mold grow indefinitely?" (Since the area of the bread (the range) is limited, the exponential growth cannot continue indefinitely. By the end of one week, more than 1 square cm will be covered and, by the end of the second week, the values of the function will be close to or will exceed the total area of the bread.)

- Students who graph using paper and pencil may decide that it makes sense to connect the points on the graph but they will not yet know how to do so. Consider stating that they are connected (in a very specific way) and their properties will be studied later.

- Students who graph using Desmos or other graphing technology will see the continuous graph. If desired, demonstrate how to use function notation in Desmos to quickly calculate output values. Typing an equation like $f(d) = 1 \cdot 2^d$ and then typing $f(2)$ or $f(-1)$ on the next line will produce the output values for those inputs.

| DO THE MATH | PLANNING NOTES |
|---|---|
| | |

## Activity 2: Functionally Speaking (*10 minutes*)

| **Instructional Routines:** Collect and Display (MLR2); Discussion Supports (MLR8) | |
|---|---|
| **Building On:** NC.M1.A-CED.2 | **Addressing:** NC.M1.F-IF.2; NC.M1.F-BF.1a |

Students have described and analyzed situations involving exponential change using graphs, tables, and equations. Now they revisit several of these contexts, viewing them as functions and expressing them using function language and notation.

Each situation involves two quantities, and students will need to choose one of these to be the independent variable and one to be the dependent variable. For all of these relationships, it is possible to choose *either* variable as the independent variable, but one choice gives a logarithmic function (which is out of the scope of this course) while the other gives an exponential function. In each case, however, students have previously worked with the context.

### Step 1

- Tell students that they will now revisit some previously seen situations. Ask students to read the three situations in the task, and then solicit a few ideas on why all of them can be seen as functions.

- Have students complete the activity in the same pairs as the last activity.

**Monitoring Tip:** Look for students who explicitly state the meaning of their variables, including units, and invite them to share during the discussion. For example, suppose in the second situation that $t$ represents the number of years and $v$ represents the value of the car, in dollars, after $t$ years. Then $v$ can be viewed as a function $f$ of $t$ where $f(t) = 18,000 \cdot \left(\frac{2}{3}\right)^t$. The meaning and units for both $t$ and $f(t)$ (or $v$) are vital elements to answering the question fully.

Building on the *Collect and Display* routine from Activity 1, invite students to brainstorm vocabulary and phrases that they associate with each variable of $y = a(b)^x$. Compile words and phrases that will assist students with setting up equations on a display. This may include labels such as “independent variable,” “dependent variable,” “growth factor,” “decay factor,” and “starting point,” and also specific words explored in previous examples in order to incorporate prior knowledge. If referencing the moldy bread scenario, the labels of “output” and “input” can also be paired with “area” and “time,” respectively.

***Advancing Student Thinking*:** Students may confuse the terms "independent" and "dependent." Help them to think about which variable depends on the other in context.

### Student Task Statement

Here are some situations we have seen previously. For each situation:

- Write a sentence of the form "____ is a function of _____."
- Indicate which is the independent and which is the dependent variable.
- Write an equation that represents the situation using function notation.

1. In a biology lab, a population of 50 bacteria reproduce by splitting. Every hour, on the hour, each bacterium splits into two bacteria.
2. Every year after a new car is purchased, it loses $\frac{1}{3}$ of its value. Let's say that the new car costs $18,000.
3. In order to control an algae bloom in a lake, scientists introduce some treatment products. The day they begin treatment, the area covered by algae is 240 square yards. Each day since the treatment began, a third of the previous day's area (in square yards) remains covered by algae. Time $t$ is measured in days.

### Step 2

- Invite selected students to present their equations, making sure to indicate what each variable represents, as well as the units of the variable.
- After each student shares, use the following sentence frames as a *Discussion Support* to help students respond: "I agree because…." or "I disagree because…." To amplify academic language, revoice student ideas by restating a statement as a question in order to clarify, apply appropriate language, and involve more students.
- For the third question, point out that it is a short step from an equation for the area covered by the algae, $A = 240 \cdot (\frac{1}{3})^t$, to a similar equation with function notation. The equation gives the area covered by the algae at each time $t$, so a function $f$ can be defined using the same expression: $f(t) = 240 \cdot (\frac{1}{3})^t$ and $A = f(t)$.

| DO THE MATH | PLANNING NOTES |
|---|---|
| | |

## Lesson Debrief (*5 minutes*)

The purpose of this lesson is to emphasize that many of the exponential growth situations students have seen in this unit can also be viewed as functions.

Use the following scenario and questions to facilitate a whole-class discussion. If necessary, choose which questions to prioritize.

Consider a bacteria population $p$, described by the equation $p = 1,000 \cdot 2^t$, where $t$ is the number of hours after it is first measured.

- "Complete the sentence for this situation: _____ is a function of _____." (The number of bacteria is a function of time, or $p$ is a function of $t$.)
- "How is $p$ related to the function $f$ given by $f(t) = 1,000 \cdot 2^t$?" ($p = f(t)$)
- "How are these equations like or different from the equations you've written previously, without function notation?" (They express the relationships the same way. Both equations can be used to produce a table or graph or to answer questions about the bacteria population. The notation $p = f(t)$ makes explicit that $p$ depends on $t$.)
- "How are the exponential functions here like or different from linear functions we saw earlier in the course?" (Both functions represent relationships where one quantity is determined by another quantity, and there is only one possible output for every input. They are different in that linear functions grow by addition and exponential functions grow by multiplication.)

**PLANNING NOTES**

### Student Lesson Summary and Glossary

The situations we have looked at that are characterized by exponential change can be seen as functions. In each situation, there is a quantity (an independent variable) that determines another quantity (a dependent variable). They are functions because any value of the independent variable corresponds to one and only one value of the dependent variable. Functions that describe exponential change are called **exponential functions**.

**Exponential function:** A function that has a constant multiplier. Another way to say this is that it changes by equal factors over equal intervals. For example, $f(t) = 100,000 \cdot (\frac{1}{5})^t$ defines an exponential function. Any time $t$ increases by 1, $f(t)$ is multiplied by a factor of $\frac{1}{5}$.

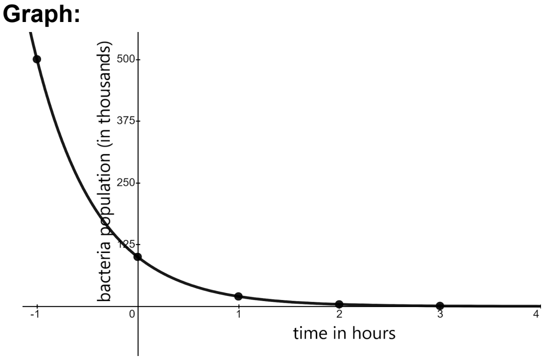
Functions can be represented by tables, graphs, equations, and descriptions.

For example, suppose $t$ represents time in hours, and $p$ is a bacteria population $t$ hours after the bacteria population was measured. For each time $t$, there is only one value for the corresponding number of bacteria, so we can say that $p$ is a function of $t$ and we can write this as $p = f(t)$.

**Table:**

| t | f(t) |
|---|---|
| -1 | 500,000 |
| 0 | 100,000 |
| 1 | 20,000 |
| 2 | 4,000 |
| 3 | 800 |

**Graph:**



| | |
|---|---|
| **Equation:** $f(t) = 100,000 \cdot (\frac{1}{5})^t$<br><br>Notice the expression in the form of $a \cdot b^t$ (on the right side of the equation) is the same as in previous equations we wrote to represent situations characterized by exponential change. | **Description:** There were 100,000 bacteria at the time the population was initially measured and the population decreases so that $\frac{1}{5}$ of it remains after each passing hour. |

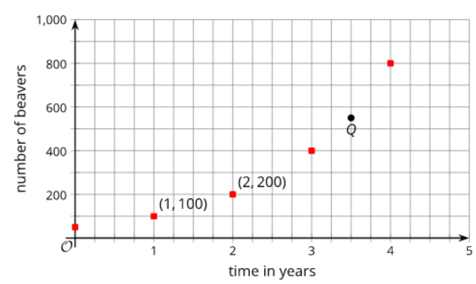
## Cool-down: Beaver Population (*5 minutes*)

**Addressing:** NC.M1.F-BF.1a

**Cool-down Guidance:** Points to Emphasize
If students struggle with function notation, there will be more chances throughout the unit to strengthen their understanding, so there is no need to slow down. However, if students are misinterpreting the scales on the graph, spend a few minutes in the next lesson, perhaps during Activity 2, eliciting strategies for correctly interpreting axes.

## Cool-down

The graph shows the population of beavers in a forest for different numbers of years after 1995. The beaver population is growing exponentially.




1. Explain why we can think of the beaver population as a function of time in years.
2. What is the meaning of the point labeled $Q$ in this context?
3. Write an equation using function notation to represent this situation.

**Student Reflection:**

Today I struggled with _____________, and I need ________________ to be successful with this.

## DO THE MATH

## INDIVIDUAL STUDENT DATA

## SUMMARY DATA

## NEXT STEPS

## TEACHER REFLECTION

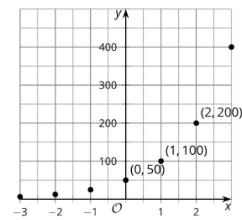
What went well in the lesson? What would you do differently next time? What happened today that will influence the planning of future lessons?

**Which math ideas from today's lesson did students grapple with most? Did this surprise you or was this what you expected?**

## Practice Problems

1. For an experiment, a scientist designs a can, 20 cm in height, that can hold water. A tube is installed at the bottom of the can allowing water to drain out. At the beginning of the experiment, the can is full. When the experiment starts, the water begins to drain, and the height of the water in the can decreases by a factor of $\frac{1}{3}$ each minute.
   a. Explain why the height of the water in the can is a function of time.
   b. The height, $h$, in cm, is a function $f$ of time $t$ in minutes since the beginning of the experiment, $h = f(t)$. Find an expression for $f(t)$.
   c. Find and record the values for $f$ when $t$ is 0, 1, 2, and 3.
   d. Find $f(4)$. What does $f(4)$ represent?
   e. Sketch a graph of $f$ by hand or use graphing technology.
   f. What happens to the level of water in the can as time continues to elapse? How do you see this in the graph?
2. A scientist measures the height, $h$, of a tree each month, and $m$ is the number of months since the scientist first measured the height of the tree.
   a. Is the height, $h$, a function of the month, $m$? Explain how you know.
   b. Is the month, $m$, a function of the height, $h$? Explain how you know.
3. A bacteria population is $10,000$. It triples each day.
   a. Explain why the bacteria population, $b$, is a function of the number of days, $d$, since it was measured to be $10,000$.
   b. Which variable is the independent variable in this situation?
   c. Write an equation relating $b$ and $d$.
4.
   a. Is the position, $p$, of the minute hand on a clock a function of the time, $t$?
   b. Is the time, $t$, a function of the position of the minute hand on a clock?
5. The area covered by a city is 20 square miles. The area grows by a factor of $1.1$ each year since it was 20 square miles.
   a. Explain why the area, $a$, covered by the city, in square miles, is a function of $t$, the number of years since its area was 20 square miles.
   b. Write an equation for $a$ in terms of $t$.

6. The graph shows an exponential relationship between $x$ and $y$.




   a. Write an equation representing this relationship.

   b. What is the value of $y$ when $x = -1$? Label this point on the graph.

   c. What is the value of $y$ when $x = -2$? Label this point on the graph.

   (From Unit 6, Lesson 7)

7. Here are two expressions:

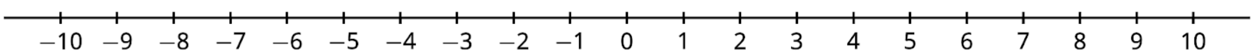
   - $x^2 \cdot x^2$
   - $(x^2)^2$

   Is the value of the first expression greater than, less than, or equal to the value of the second expression? How do you know?

   (From Unit 6, Lessons 1 and 2)

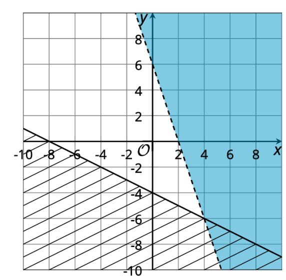
8. Here is an inequality: $3x + 1 > 34 - 4x$.

   Graph the solution set to the inequality on the number line.




   (From Unit 2)

9. Two inequalities are graphed on the same coordinate plane.




   Select **all** of the points that are solutions to the system of the two inequalities.

   a. $(4, -6)$
   b. $(4, 6)$
   c. $(-4, -6)$
   d. $(-4, 6)$
   e. $(6, -8)$
   f. $(7, -9)$
   g. $(-8, 6)$

   (From Unit 2)

10.

   a. The distance around a circle, the circumference, $C$, is given as a function of the diameter, $d$, based on the rule $C = \pi d$. If a circle has a diameter of 6.5 inches, how long is the circumference?

   b. The volume of a sphere, $V$, is given as a function of the radius, $r$, based on the rule $V = \frac{4}{3}\pi r^3$. If a sphere has a radius of 3.25 inches, what is the volume of the sphere?

   (Addressing NC.7.G.4 and NC.8.F.9)

# Lesson 12: Interpreting Exponential Functions

## PREPARATION

| Lesson Goals | Learning Targets |
|---|---|
| • Determine whether a graph that represents a situation should be continuous or discrete.<br>• Interpret graphs of exponential functions and equations written in function notation to answer questions about a context.<br>• Use function notation to describe an exponential relationship represented by a graph.<br>• Use graphing technology to graph exponential functions and analyze their domains. | • I can analyze a situation and determine whether it makes sense to connect the points on the graph that represents the situation.<br>• When I see a graph of an exponential function, I can make sense of and describe the relationship using function notation.<br>• I can use graphing technology to graph exponential functions and analyze their domains. |

### Lesson Narrative

In this lesson, students study exponential functions and their graphs in context. Given a graph or a description of a relationship, they write one quantity as a function of another and then use the function to answer questions about the context. They also produce graphs of exponential functions, paying close attention to the appropriate domain and range, which are both restricted by the context (MP2 and MP6).

**What is the main purpose of this lesson? What is the one thing you want your students to take away from this lesson?**

### Focus and Coherence

| Building On | Addressing | Building Towards |
|---|---|---|
| **NC.8.F.1:** Understand that a function is a rule that assigns to each input exactly one output.<br>• Recognize functions when graphed as the set of ordered pairs consisting of an input and exactly one corresponding output.<br>• Recognize functions given a table of values or a set of ordered pairs.<br><br>(*continued*) | **NC.M1.F-IF.2:** Use function notation to evaluate linear, quadratic, and exponential functions for inputs in their domains, and interpret statements that use function notation in terms of a context.<br><br>**NC.M1.F-IF.5:** Interpret a function in terms of the context by relating its domain and range to its graph and, where applicable, to the quantitative relationship it describes. | **NC.M1.A-CED.1:** Create equations and inequalities in one variable that represent linear, exponential, and quadratic relationships and use them to solve problems. |

| | | |
|---|---|---|
| **NC.8.F.5:** Qualitatively analyze the functional relationship between two quantities.<br>● Analyze a graph determining where the function is increasing or decreasing; linear or non-linear.<br>● Sketch a graph that exhibits the qualitative features of a real-world function. | **NC.M1.F-IF.7:** Analyze linear, exponential, and quadratic functions by generating different representations, by hand in simple cases and using technology for more complicated cases, to show key features, including: domain and range; rate of change; intercepts; intervals where the function is increasing, decreasing, positive, or negative; maximums and minimums; and end behavior. | |
| **NC.M1.A-REI.11:** Build an understanding of why the $x$-coordinates of the points where the graphs of two linear, exponential, and/or quadratic equations $y=f(x)$ and $y=g(x)$ intersect are the solutions of the equation $f(x)=g(x)$ and approximate solutions using graphing technology or successive approximations with a table of values. | **NC.M1.F-BF.1a:** Build linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two ordered pairs (include reading these from a table).<br><br>**NC.M1.A-SSE.1a:** Identify and interpret parts of a linear, exponential, or quadratic expression, including terms, factors, coefficients, and exponents. | |

## Agenda, Materials, and Preparation

Technology is required for this lesson in the Warm-up and Activity 2: Acquire devices that can access Desmos (recommended) or other graphing technology. It is ideal if each student has their own device.

- **Bridge** (*Optional, 5 minutes*)
- **Warm-up** (*5 minutes*)
- **Activity 1** (*10 minutes*)
- **Activity 2** (*15 minutes*)
  - Blank paper (consider one sheet per student, or to save paper, one sheet for every two students)
- **Lesson Debrief** (*5 minutes*)
- **Cool-down** (*5 minutes*)
  - M1.U6.L12 Cool-down (print 1 copy per student)

# LESSON

## Bridge (*Optional, 5 minutes*)

| **Building On:** NC.8.F.1; NC.8.F.5 | **Building Towards:** NC.M1.F-IF.5 |
|---|---|

The purpose of this bridge is to elicit the idea that graphs can be discrete or continuous based on the context, which will be useful later in this lesson when students interpret exponential functions and make sense of whether the graph should be discrete or continuous. When students pay close attention to the appropriate domain and range, both restricted by the context, they are engaging in work that is important in modeling with mathematics (MP4).

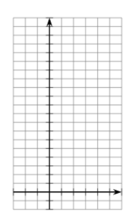
In order to find the table of values, students may choose to use graphing technology strategically (MP5), if available.

### Student Task Statement

Here are descriptions of relationships between quantities.

1. A cab charges $\$1.50$ per mile plus $\$3.50$ for entering the cab. The cost of the ride is a function of the miles, $m$, ridden and is defined by $c(m) = 1.50m + 3.50$.
   a. Make a table of at least five pairs of values that represent the relationship.
   b. Plot the points. Label the axes of the graph.
   c. Should the points be connected? Are there any input or output values that don't make sense? Explain.

| $m$ | $c$ |
|---|---|
| | |
| | |
| | |
| | |
| | |



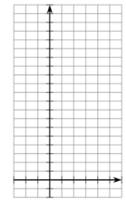
2. The admission to the state park is $\$5.00$ per vehicle plus $\$1.50$ per passenger. The total admission for one vehicle is a function of the number of passengers, $p$, defined by the equation $a(p) = 5 + 1.50p$.

   a. Make a table of at least five pairs of values that represent the relationship.

   b. Plot the points. Label the axes of the graph.

   c. Should the points be connected? Are there any input or output values that don't make sense? Explain.

| $p$ | $a$ |
| --- | --- |
| | |
| | |
| | |
| | |
| | |



## DO THE MATH

## PLANNING NOTES

## Warm-up: Equivalent or Not? (*5 minutes*)

| **Instructional Routines:** Poll the Class; Round Robin | |
| --- | --- |
| **Building On:** NC.M1.A-REI.11 | **Addressing:** NC.M1.A-SSE.1a |

This warm-up addresses a common confusion among students mistaking the expression $2^x$ for the expression $x^2$. One good way to address this confusion is by evaluating the expressions for different values of $x$, which is the path students are likely to take here. Another way to illustrate the distinction is by graphing, though students are less likely to take this approach.

### Step 1

- Display the task for all to see and ask students to read it quietly to themselves.
- Ask what they think Lin means by "equivalent." They may remember from earlier grades, or they may guess based on contrasting her statement with Diego's statement. Before they start working, be sure that students understand "equivalent" to mean "equal" no matter what value is used in place of $x$.
- Give students a minute of independent think time, then *Poll the Class* for agreement with Lin, Diego, or neither of them.
- Have students share their reasoning with a partner using the *Round Robin* routine.

### Student Task Statement

Lin and Diego are discussing two expressions: $x^2$ and $2^x$.

- Lin says, "I think the two expressions are equivalent."
- Diego says, "I think the two expressions are only equal for *some* values of $x$."

Do you agree with either of them? Explain or show your reasoning.

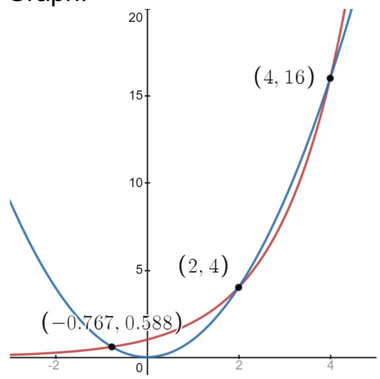
### Step 2

- Consider using a completed table and a graph, as shown below, to facilitate discussion.

Completed table:

| $x$ | $x^2$ | $2^x$ |
|---|---|---|
| 1 | 1 | 2 |
| 2 | 4 | 4 |
| 3 | 9 | 8 |
| 4 | 16 | 16 |
| 5 | 25 | 32 |
| 10 | 100 | 1024 |

Graph:




- Possible questions for discussion:
  - “For what values of $x$ are the two expressions equal?” (when $x$ is 2, when $x$ is 4, and also when $x$ is approximately -0.767 as seen when examining the graphs of the two equations on the same coordinate plane)
  - “Besides substituting different values of $x$, are there other ways to tell if the two expressions are equal for all values of $x$ or only some values of $x$?” (One way might be to think about possible values the expressions could take when $x$ is an odd number. $2^x$ means 2 multiplied by itself $x$ times. When $x$ is an odd number, the value of $2^x$ will be even because of the multiplication by 2. But multiplying an odd-number $x$ by itself, as in $x^2$, will result in an odd number, so we know that in those instances the expressions are not equal. Another way might be to graph the equations $y = x^2$ and $y = 2^x$ on the same coordinate plane and examine the graphs. This method allows for us to see a third value of x $x$ where the two expressions are equal, -0.767. )
  - “Which expression grows more quickly?” (For positive values of $x$, $2^x$ grows much more quickly than $x^2$.)

| DO THE MATH | PLANNING NOTES |
|---|---|
| | |

### Activity 1: Cost of Solar Cells (*10 minutes*)

| **Instructional Routine:** Three Reads (MLR6) - Responsive Strategy | |
|---|---|
| **Addressing:** NC.M1.F-IF.2 | **Building Towards:** NC.M1.A-CED.1 |

Previously, students were presented with descriptions of functions and, in one case, an equation that represents a function. In this activity, students are given the graph of a function and asked to analyze the underlying relationship and describe it using function notation.

The data points used in this task are approximate, but the costs of solar energy during the time period mentioned in this activity can be appropriately modeled by an exponential decay model.

### Step 1

- Elicit what students already know about solar cells. If students are unfamiliar with solar power and units of measurement for power, give a brief introduction. Solar cells turn energy from the sun into electricity, a form of energy that is useful to humans, and use this electricity as a power source. Power is measured using units called watts. Over the course of the past several decades, the cost of solar cells has decreased. That is, we can manufacture solar cells that generate the same amount of electricity for less money.
- Read the opening sentence of the task and display the graph for all to see. To help them process the information in the graph and think about functions, ask if the price per watt is a function of time and if the time is a function of the price per watt.

### Step 2

- Ask students to arrange themselves in groups of two or use visibly random grouping.
- Provide students with 4–5 minutes of quiet work time on the task before working with a partner to complete the questions.

**RESPONSIVE STRATEGY**

Use this routine to familiarize students with the context. Read the opening sentences of the task and display the graph for all to see, then ask students,"What is this situation about?" (installing solar panels, changing cost of installing solar panels). As needed, share the information in the first bullet in Step 1. Read the opening sentences aloud again and ask students, "What can be counted or measured?" For this second read, ask students to focus on the quantities being measured or counted in the context (cost of installing solar panels, watts, years since 1977), rather than specific given values. After a third read (aloud or individual), share the first two questions and ask students to brainstorm strategies to answer the question.

Three Reads (MLR6)

**RESPONSIVE STRATEGY**

Demonstrate and encourage students to use color coding and annotation to highlight connections between representations in a problem. For example, highlighting f(4) as written in the question and then highlighting the corresponding output value on the graph. Some students may benefit from reviewing that the x-coordinate is 4 and f(4) is the y-coordinate.

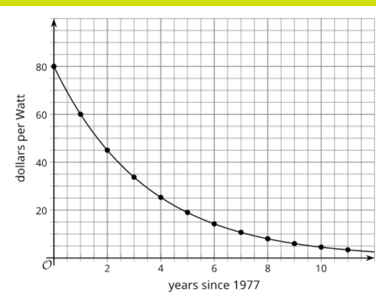
Supports accessibility for: Visual-spatial processing

***Advancing Student Thinking***: If students struggle with the function notation in the questions, ask them to recall what each part of what "$f(t)$" means, or remind them that the $f$ is the name of the function, and the $t$ is the input value.

### Student Task Statement

The cost, in dollars, to install solar panels that produce 1 watt of solar power is a function of the number of years since 1977, $t$.

From 1977 to 1987, the cost could be modeled by an exponential function $f$. Here is the graph of the function.




1. What is the statement $f(9) \approx 6$ saying about this situation?
2. What is $f(4)$? What about $f(3.5)$? What do these values represent in this context?
3. When $f(t) = 45$, what is $t$? What does that value of $t$ represent in this context?
4. By what factor did the cost of solar cells change each year? (If you get stuck, consider creating a table.)
5. How would you interpret the statement $f(-1) = 100$? Do you think this statement is reliable?

### Step 3

- Facilitate a whole-class discussion focused on how students answered questions 1 through 3.
  - "What does the 4 in $f(4) = 25$ mean?" (4 years after 1977, which is 1981)
  - "What does $f(2) = 45$ mean?" (2 years after 1977, 1979, it cost $45 to install 1 watt of solar energy.)
  - Interpreting statements such as $f(2) = 45$ in terms of the context is a chance to highlight precision in language (MP6). Ask students questions such as:
    - "How did you determine whether you had been given an input, an output value, or both in each question?"
    - "When using the graph, how did you use a given input value to find the output?"
    - "When using the graph, how did you use a given output value to find the input?"
  - Make sure students agree that the decay factor is $\frac{3}{4}$, and if time allows, ask which points they used to calculate it.

| DO THE MATH | PLANNING NOTES |
|---|---|
| | |

## Activity 2: Paper Folding (*15 minutes*)

| **Instructional Routines:** Graph It; Aspects of Mathematical Modeling; Critique, Correct, Clarify (MLR3) | | |
|---|---|---|
| **Building On:** NC.M1.A-REI.11 | **Addressing:** NC.M1.F-IF.5; NC.M1.F-IF.7; NC.M1.F-BF.1a | **Building Towards:** NC.M1.A-CED.2 |

This *Graph It* activity prompts students to graph functions to solve problems and think about appropriate values of their variables. The context is the thickness and area of a sheet of paper that is repeatedly folded.

For both functions, only whole numbers make sense as inputs. Students consider why this makes sense in this context. (This is in contrast, for example, to the price of installing solar panels in the earlier activity. While the data points given were on an annual basis, it made sense to "connect the dots" because there is a price for installing solar panels at any given time.) Without instruction about how to create a discrete graph with their graphing technology, students are likely to produce a continuous graph. This opportunity to discuss how to interpret a model for a particular context (MP4) engages students in the interpreting *Aspects of Mathematical Modeling*.

### Step 1

- Give each student a sheet of paper. Ask them to fold it in half, and in half again, as many times as they can. (To save paper, you might have students do this in pairs.) Once they have folded it as many times as they can, ask a few students to share how many times they folded it, and estimate the thickness of the folded paper. Tell students that they are going to investigate the relationship between the number of folds and the thickness of the paper.
- Provide access to graphing technology. This activity requires students to be mindful about the scale and graphing window.

**Monitoring Tip:** Look for the different ways students use to find out how many folds are needed to reach a certain thickness. Identify students who used a table, a graph, or an equation so they can share later.

***Advancing Student Thinking***: If students have trouble organizing their data points, suggest that they use a table.

### Student Task Statement

1. The thickness, $t$, in millimeters, of a folded sheet of paper after it is folded $n$ times is given by the function $t(n) = (0.05) \cdot 2^n$.
   a. What does the number $0.05$ represent in the rule of the function?
   b. Use graphing technology to graph the function $t(n) = (0.05) \cdot 2^n$.
   c. How many folds does it take before the folded sheet of paper is more than 1 mm thick? How many folds before it is more than 1 cm thick? Explain how you know.
2. The area of a sheet of paper is 93.5 square inches.
   a. Find the remaining visible area of the sheet of paper after it is folded in half once, twice, and three times.
   b. Write a function rule expressing the visible area, $a$, of the sheet of paper in terms of the number of times it has been folded, $n$.
   c. Use graphing technology to graph the function.
   d. In this context, can $n$ take negative values? Explain your reasoning.
   e. Can $a$ take negative values? Explain your reasoning.

### Are You Ready For More?

1. Using the model in this task, how many folds would be needed to get 1 meter in thickness? 1 kilometer in thickness?
2. Do some research: what is the current world record for the number of times humans were able to fold a sheet of paper?

### Step 2

- Facilitate a whole-class discussion focused on:
  - How students can identify the growth or decay factor, given an equation or a description.
  - How students chose a reasonable window for graphing this situation.

### Step 3

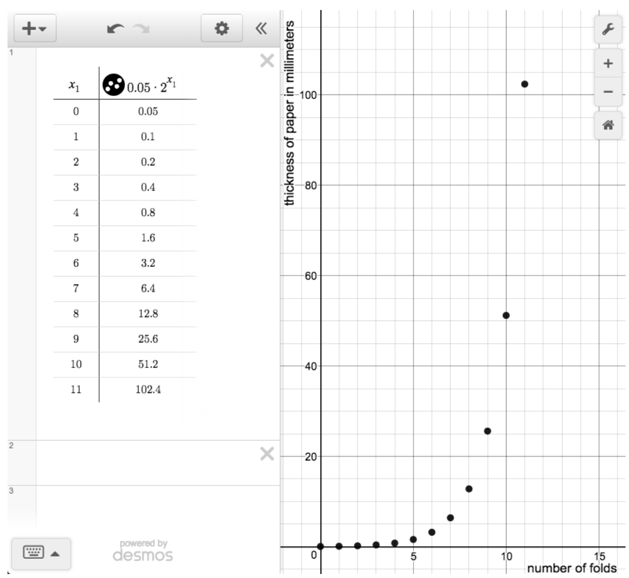
- Using a graphing window suggested by students, use Desmos to display a graph of the function $t(n) = (0.05) \cdot 2^n$. Have students make suggestions about refining the window if necessary.
- Model for students how to enter an output value, such as $y = 1$, into Desmos to find the $x$ value where $y = 1$ and $t(n) = (0.05) * 2^n$ intersect.

Use the *Critique, Correct, Clarify* routine by presenting an incorrect/ambiguous “first draft” response to the question, “How many folds does it take before the folded sheet of paper is more than 1 cm thick?” Consider using this first draft response: “It takes 4.5 folds because I looked at the line and the graph.”

- Ask students to first identify the error and any unclear or incomplete parts of the response.
- Ask two or three students to share their ideas and annotate the response for all to see as students share, in order to indicate the parts of the response that could use improvement. (If needed, call students' attention to the fact that a non-whole number of folds does not make sense, that the response does not tell us what information from the graph was relevant for answering the question, and it is not clear what "the line" is. Publicly annotate by underlining or circling these three parts of the response.)
- Then ask students to work with a partner to write an improved explanation. (Sample improved explanation: It takes 5 folds because the graph and the table include the point (4,8) and the point (5,16), which means the paper was 8 mm thick at 4 folds and 16 mm thick at 5 folds.)
- Invite one or two students to share their improved explanations with the class, and scribe as they share, for all to see. Invite the class to suggest additional wording as you scribe, including both further edits and additional ideas. This final step generates a "third draft" for the whole class, and helps students evaluate, and improve on, the written mathematical arguments of others, as they clarify their understanding of discrete graphs.

### Step 4

- Ask students what other restrictions there are on the variables in the two situations. Remind them that allowable inputs to the function form the *domain*, while allowable outputs form the *range*.
  - In both situations, the number of folds must be a whole number. Also, practically speaking, a large number of folds, such as 36, is not possible. So the domain of each function is something like whole numbers between 0 and 6. (Students can experiment to see how many folds are possible.)
  - The thickness and area of the paper are not restricted to whole numbers, but they are also limited by the number of folds. So the range of each function is also discrete, and it has an upper limit.
- Time permitting, you might demonstrate how to graph only whole number values of $n$. In Desmos, this is done by creating a table using the variable and the rule as headings, and then manually entering values of the variable. Here is an illustration:




**DO THE MATH**

**PLANNING NOTES**

## Lesson Debrief (*5 minutes*)

The purpose of this lesson is to use exponential functions and function notation to answer questions about different situations.

Choose what questions to focus the discussion on, whether students should first have an opportunity to reflect in their workbooks or talk through these with a partner, and what questions will be prioritized in the full class discussion.

In a previous lesson, we saw that a function $f$ defined by $f(t) = 240 \cdot (\frac{1}{3})^t$ represents the area of algae in square yards after $t$ weeks of treatment. Discuss how the equation could be used to:

- produce a graph (Use Desmos or other graphing technology, making sure to set the window so that the maximum area, 240 square yards, is visible.)
- find the area of the algae 6 weeks after treatment (Evaluate $f(6)$, or use graphing technology to trace the graph to find the other coordinate when $t = 6$.)
- find out when the algae would drop below 10 square yards (Graph the function, along with the equation $y = 10$, and find the intersection.)

Ask students what $f(1) = 80$ and $f(2) = \frac{80}{3}$ mean in this context. To reiterate the importance of choosing an appropriate graphing window, ask students to give examples of a graphing window that would be appropriate (or inappropriate) for this function. Time permitting, demonstrate the outcomes of their suggestions using graphing technology.

**PLANNING NOTES**

### Student Lesson Summary and Glossary

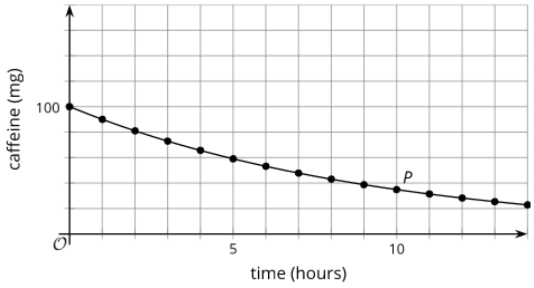
Earlier, we used equations to represent situations characterized by exponential change. For example, to describe the amount of caffeine $c$ in a person's body $t$ hours after an initial measurement of 100 mg, we used the equation $c = 100 \cdot (\frac{9}{10})^t$.

Notice that the amount of caffeine is a *function* of time, so another way to express this relationship is $c = f(t)$ where $f$ is the function given by $f(t) = 100 \cdot (\frac{9}{10})^t$.

We can use this function to analyze the amount of caffeine. For example, when $t$ is 3, the amount of caffeine in the body is $100 \cdot (\frac{9}{10})^3$ or $100 \cdot \frac{729}{1{,}000}$, which is 72.9. The statement $f(3) = 72.9$ means that 72.9 mg of caffeine are present 3 hours after the initial measurement.

We can also graph the function $f$ to better understand what is happening. The point labeled $P$, for example, has coordinates approximately $(10, 35)$ so it takes about 10 hours after the initial measurement for the caffeine level to decrease to 35 mg.

A graph can also help us think about the values in the domain and range of a function. Because the body breaks down caffeine continuously over time, the domain of the function—the time in hours—can include non-whole numbers (for example, we can find the caffeine level when $t$ is 3.5). In this situation, negative values for the domain would represent the time *before* the initial measurement. For example $f(-1)$ would represent the amount of caffeine in the person's body 1 hour before the initial measurement. The range of this function would not include negative values, as a negative amount of caffeine does not make sense in this situation.



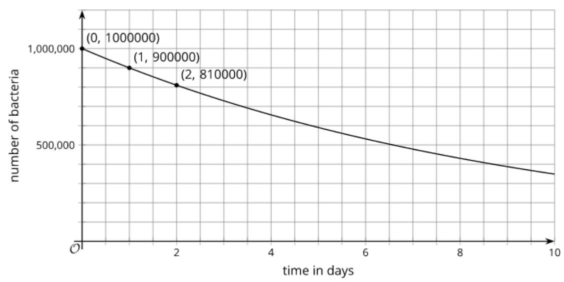
## Cool-down: Bacteria Population (*5 minutes*)

| **Addressing:** NC.M1.F-IF.2 | **Building Towards:** NC.M1.A-CED.1 |
|---|---|
| **Cool-down Guidance:** Points to Emphasize<br>Students will have more opportunities to interpret graphs, so use the results from this cool-down to choose points to emphasize in subsequent lessons and highlight and reinforce strategies for identifying points on graphs. | |

In the case that students are unfamiliar with antibiotics, explain that they kill bacteria and keep bacteria from reproducing.

### Cool-down

The graph shows the bacteria population on a petri dish as a function $f$ the days $d$ since an antibiotic is introduced.




1. What is the approximate value of $f(4.5)$?
2. Approximately what is $d$ when $f(d) = 400,000$?
3. Explain what you would do, using your usual graphing technology, to be able to see $f(15)$ on the graph.

**Student Reflection:**

Your friend asks, "How are you so good at math?" How do you respond? What advice would you give your friend to improve?

### DO THE MATH

### INDIVIDUAL STUDENT DATA

### SUMMARY DATA

**NEXT STEPS**

## TEACHER REFLECTION

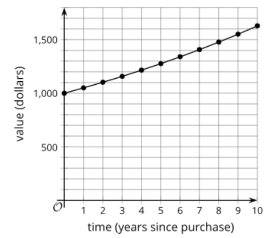
**What went well in the lesson? What would you do differently next time? What happened today that will influence the planning of future lessons?**

**Today's cool-down offered students a chance to give advice on how to improve in math. What was the best advice you saw given? How can you share that with your students?**

## Practice Problems

1. The number of people with the flu during an epidemic is a function, $f$, of the number of days, $d$, since the epidemic began. The equation $f(d) = 50 \cdot (\frac{3}{2})^d$ defines $f$.
   a. How many people had the flu at the beginning of the epidemic? Explain how you know.
   b. How quickly is the flu spreading? Explain how you can tell from the equation.
   c. What does $f(1)$ mean in this situation?
   d. Does $f(3.5)$ make sense in this situation?

2. The function $f$ gives the dollar value of a bond $t$ years after the bond was purchased. The graph of $f$ is shown.




   a. What is $f(0)$? What does it mean in this situation?
   b. What is $f(4.5)$? What does it mean in this situation?
   c. When is $f(t) = 1500$? What does this mean in this situation?

3. (*Technology required.*) A function $f$ gives the number of stray cats in a town $t$ years since the town started an animal control program. The program includes both sterilizing stray cats and finding homes to adopt them. An equation representing $f$ is $f(t) = 243(\frac{1}{3})^t$.
   a. What is the value of $f(t)$ when $t$ is 0? Explain what this value means in this situation.
   b. What is the approximate value of $f(t)$ when $t$ is 0.5? Explain what this value means in this situation.
   c. What does the number $\frac{1}{3}$ tell you about the stray cat population?
   d. Use technology to graph $f$ for values of $t$ between 0 and 4. What graphing window allows you to see values of $f(t)$ that correspond to these values of $t$?

4. (*Technology required.*) Function $g$ gives the amount of a chemical in a person's body, in milligrams, $t$ hours since the patient took the drug. The equation $g(t) = 600 \cdot (\frac{3}{5})^t$ defines this function.
   a. What does the fraction $\frac{3}{5}$ mean in this situation?
   b. Use graphing technology to create a graph of $g$.
   c. What are the domain and range of $g$? Explain what they mean in this situation.

5. The dollar value of a moped is a function of the number of years, $t$, since the moped was purchased. The function $f$ is defined by the equation $f(t) = 2,500 \cdot (\frac{1}{2})^t$.

   What is the **best** choice of domain for the function $f$?

   a. $-10 \le t \le 10$
   b. $-10 \le t \le 0$
   c. $0 \le t \le 10$
   d. $0 \le t \le 100$

6. All of the students in a classroom list their birthdays.
   a. Is the birthdate, $b$, a function of the student, $s$? Explain your reasoning.
   b. Is the student, $s$, a function of the birthdate, $b$? Explain your reasoning.

   (From Unit 6, Lesson 11)

7. The trees in a forest are suffering from a disease. The population of trees, $p$, in thousands, is modeled by the equation $p = 90 \cdot \left(\frac{3}{4}\right)^t$, where $t$ is the number of years since 2000.

   a. What was the tree population in 2001? What about in 1999?

   b. What does the number $\frac{3}{4}$ in the equation for $p$ tell you about the population?

   c. What is the last year when the population was more than 250,000? Explain how you know.

   (From Unit 6, Lesson 7)

8. A patient receives 1,000 mg of a medicine. Each hour, $\frac{1}{5}$ of the medicine in the patient's body decays.

   a. Complete the table with the amount of medicine in the patient's body.

   b. Write an equation representing the number of mg of the medicine, $m$, in the patient's body $h$ hours after receiving the medicine.

   c. Use your equation to find $m$ when $h = 10$. What does this mean in terms of the medicine?

   (From Unit 6, Lesson 6)

| Hours since receiving medicine | mg of medicine left in body |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| $h$ | |

9. Rewrite each expression using the fewest number of exponents.

   a. $3x^a \cdot x^b$

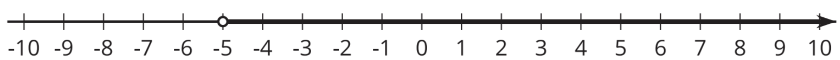
   b. $\frac{-5y^5 \cdot 4y^{-2} \cdot y^4}{y^{-2}}$

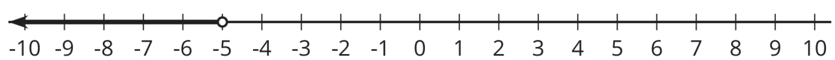
   c. $\frac{(5n^2)^4}{-5n^2}$

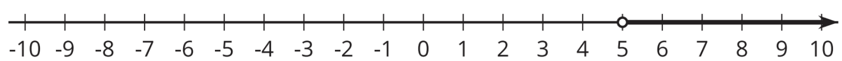
   (From Unit 6, Lesson 2)

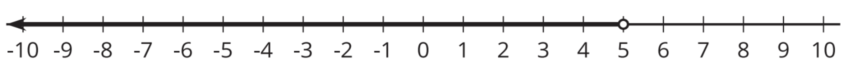
10. Mai wants to graph the solution to the inequality $5x - 4 > 2x - 19$ on a number line. She solves the equation $5x - 4 = 2x - 19$ for $x$ and gets $x = -5$.

    Which graph shows the solution to the inequality?

    a. 

    b. 

    c. 

    d. 

    (From Unit 2)

11. A car is traveling on a small highway and is either going 55 miles per hour or 35 miles per hour, depending on the speed limits, until it reaches its destination 200 miles away. Letting $x$ represent the amount of time in hours that the car is going 55 miles per hour, and $y$ being the time in hours that the car is going 35 miles per hour, an equation describing the relationship is: $55x + 35y = 200$.

    a. If the car spends 2.5 hours going 35 miles per hour on the trip, how long does it spend going 55 miles per hour?

    b. If the car spends 3 hours going 55 miles per hour on the trip, how long does it spend going 35 miles per hour?

    c. If the car spends no time going 35 miles per hour, how long would the trip take? Explain your reasoning.

    (Addressing NC.8.F.1 and NC.8.F.5)[1]

[1] Adapted from IM 6–8 Math https://curriculum.illustrativemathematics.org/MS/index.html, which was originally developed by Open Up Resources and authored by Illustrative Mathematics, and is copyright 2017–2019 by Open Up Resources. It is licensed under the Creative Commons Attribution 4.0 International License (CC BY 4.0). OUR's 6–8 Math Curriculum is available at https://openupresources.org/math-curriculum/. Adaptations and updates to IM 6–8 Math are copyright 2019 by Illustrative Mathematics, and are licensed under the Creative Commons Attribution 4.0 International License (CC BY 4.0).

# Lesson 13: Modeling Exponential Behavior

## PREPARATION

| Lesson Goals | Learning Targets |
|---|---|
| • Choose an appropriate model for a situation when given data.<br>• Determine graphing windows that would make a graph more informative or meaningful.<br>• Use exponential functions to model situations that involve exponential growth or decay. | • When given data, I can determine an appropriate model for the situation described by the data.<br>• I can use exponential functions to model situations that involve exponential growth or decay. |

### Lesson Narrative

In this lesson, students study the successive bounce heights of balls, model the relationship between the number of bounces and the bounce heights, and use that model to answer questions about the ball's bounciness. There are options for how much of the modeling cycle (MP4) students undertake. In one optional activity, students collect data for the bounce heights of different balls, while in two activities, the data are provided. In all cases, the data are deliberately "messy" to mirror data students would gather through experimentation, and students are left to decide what kind of model to use. Though the data are not perfectly exponential or linear, an exponential model fits much better.

Once students decide to use an exponential model, they still have to find the parameters that best fit the data while maintaining a reasonable level of accuracy (MP6). Some may just use the quotient of the first two bounce heights, while others may look more closely at all of the bounce heights. As they compare and critique the different models, students construct viable arguments and critique the reasoning of others (MP3).

**What are you excited for your students to be able to do after this lesson?**

## Focus and Coherence

| Building On | Addressing |
|---|---|
| **NC.6.RP.4:** Use ratio reasoning to solve real-world and mathematical problems with percents by:<br>• Understanding and finding a percent of a quantity as a ratio per 100.<br>• Using equivalent ratios, such as benchmark percents (50%, 25%, 10%, 5%, 1%), to determine a part of any given quantity.<br>• Finding the whole, given a part and the percent.<br><br>**NC.M1.S-ID.6:** Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.<br>**a.** Fit a least squares regression line to linear data using technology. Use the fitted function to solve problems.<br>**b.** Assess the fit of a linear function by analyzing residuals. | **NC.M1.F-IF.5:** Interpret a function in terms of the context by relating its domain and range to its graph and, where applicable, to the quantitative relationship it describes.<br><br>**NC.M1.F-LE.1:** Identify situations that can be modeled with linear and exponential functions, and justify the most appropriate model for a situation based on the rate of change over equal intervals.<br><br>**NC.M1.F-LE.5:** Interpret the parameters $a$ and $b$ in a linear function $f(x) = ax + b$ or an exponential function $g(x) = ab^x$ in terms of a context.<br><br>**NC.M1.S-ID.6c:** Fit a function to exponential data using technology. Use the fitted function to solve problems. |

## Agenda, Materials, and Preparation

- **Bridge** (*Optional, 5 minutes*)
- **Warm-up** (*5 minutes*)
- **Activity 1** (*15 minutes*)
  - Bouncing Tennis Ball 3 video: https://bit.ly/TennisBallBounce
- **Activity 2** (*10 minutes*)
- **Activity 3** (*Optional, 40 minutes*)
  - A collection of balls that bounce
  - Measuring tapes
  - Graphing technology should be available if students request it. Acquire devices that can access Desmos (recommended) or other graphing technology.
- **Lesson Debrief** (*5 minutes*)
- **Cool-down** (*5 minutes*)
  - M1.U6.L13 Cool-down (print 1 copy per student)

## LESSON

### Bridge (*Optional, 5 minutes*)

**Building On:** NC.6.RP.4

The purpose of this bridge is to offer students an opportunity to calculate the percentage of a value, a skill developed in grade 6, to support their reasoning in Activity 1 of this lesson.

#### Student Task Statement

Order these three values from least to greatest. Explain or show your reasoning.[1]

- 65% of 80
- 82% of 50
- 170% of 30

[1] Adapted from IM 6–8 Math https://curriculum.illustrativemathematics.org/MS/index.html, which was originally developed by Open Up Resources and authored by Illustrative Mathematics, and is copyright 2017–2019 by Open Up Resources. It is licensed under the Creative Commons Attribution 4.0 International License (CC BY 4.0). OUR's 6–8 Math Curriculum is available at https://openupresources.org/math-curriculum/. Adaptations and updates to IM 6–8 Math are copyright 2019 by Illustrative Mathematics, and are licensed under the Creative Commons Attribution 4.0 International License (CC BY 4.0).

**DO THE MATH**

**PLANNING NOTES**

## Warm-up: Wondering about Windows (*5 minutes*)

**Building Towards:** NC.M1.F-IF.4

This warm-up prompts students to think about how changing the graphing window influences what you can see and understand about an exponential function. As students work to manipulate the axes of the graphing window to showcase key features of interest, they demonstrate using tools strategically (MP5).

**RESPONSIVE STRATEGY**

Provide students with graphing technology to explore different graphing windows.

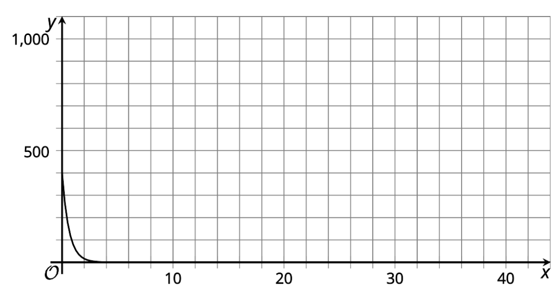
Supports accessibility for: Conceptual processing; Visual-spatial processing

### Step 1

- Invite students to share some suggestions for a graphing window that are more helpful and those that are less helpful. Ask them to explain their reasoning.

### Student Task Statement

Here is a graph of a function $f$ defined by $f(x) = 400 \cdot (0.2)^x$.




1. Identify the approximate graphing window shown.
2. Suggest a new graphing window that would:
   a. make the graph more informative or meaningful
   b. make the graph less informative or meaningful

Be prepared to explain your reasoning.

### Step 2

- Help students understand that, as a general rule, the $x$-values to show for a graph are usually determined by the quantity we are interested in studying. The $y$-values need to be selected carefully so that:
  - Data of interest show up on the graph.
  - Interesting trends (of increase or decrease) are as visible as possible.
- If time permits, discuss:
  - "Why does the graph show such a sharp decrease between $x$-values of 0 and 2 and then start to flatten out?" (The decay factor is 0.2, which means that whenever $x$ increases by 1, it loses 0.8 of its quantity and keeps only 0.2. That decay is more apparent when the quantity is larger. As it gets smaller, and given the scale of the graph, it is harder to see the change. For example, when $x$ increases from 0 to 1, $f(x)$ decreases by 320, because $(0.8) \cdot 400 = 320$. But when $x$ increases from 2 to 3, $f(x)$ decreases by $(0.8) \cdot 16$ or 10.8, which is a much smaller drop.)

| DO THE MATH | PLANNING NOTES |
|---|---|
| | |

## Activity 1: Beholding Bounces (*15 minutes*)

| **Instructional Routines:** Graph It; Aspects of Mathematical Modeling; Compare and Connect (MLR7) - Responsive Strategy | |
|---|---|
| **Building On:** NC.M1.S-ID.6 | **Addressing:** NC.M1.F-LE.1; NC.M1.F-LE.5; NC.M1.S-ID.6c |

In this *Graph It* activity, students examine the successive heights that a tennis ball reaches after several bounces on a hard surface and consider how to model the relationship between the number of bounces and the height of the rebound. To do so, they need to determine the growth factor of successive bounce heights. Because some data points are provided in the activity rather than students gathering the data, students engage in only some aspects of mathematical modeling. To engage students in the full *Aspects of Mathematical Modeling* cycle that includes data gathering, consider asking students to measure the bounce heights of a ball, as suggested in the optional Activity 3. Note: in question 4, it is suggested to graph the data in “log mode” in Desmos. This makes for better fits of data that do not follow a linear pattern and has the benefit of providing comparable results to other technology. Making graphing technology available gives students an opportunity to choose appropriate tools strategically (MP5).

### Step 1

- Play the video, “Bouncing Tennis Ball 3,” found at this link: https://bit.ly/TennisBallBounce. Ask students how they might go about collecting data for a ball bouncing, similar to this one. Consider replaying the video a few times to elicit more responses.
- Ask students to arrange themselves in small groups or use visibly random grouping, and ask them to work together to answer the activity’s questions.

**RESPONSIVE STRATEGY**

Highlight connections between representations by using color or annotations to connect the numbers in the table to the symbols in the equation.

Supports accessibility for: Conceptual processing; Visual-spatial processing

**Monitoring Tip:** Real-world data are often messy, and that is the case for the data provided here. Monitor for students who try the following approaches in deciding whether a linear or exponential model is more appropriate for modeling the data:

- Use the table of values and look at successive differences.
- Use the table of values and look at successive quotients.
- Plot the points and observe the general trend in bounce height.
- Plot the points using graphing technology and use the technology to generate a line or curve of best fit.

While each successive bounce height is about half of the preceding height, there is variation in the data, with the largest factor being a little more than 0.55 and the smallest a little less than 0.47. Monitor the way students choose to deal with this variation, which affects the model they consider appropriate. They may:

- Decide that an exponential model is not appropriate because the growth factor is different from bounce to bounce.
- Make a rough approximation for the decay factor; for example, observe that each bounce height $h$ is about half the preceding bounce height: $h = 75 \cdot (\frac{1}{2})^n$.
- Find and use the decay factor from the first two points: $h = 75 \cdot (\frac{8}{15})^n$.
- Take an average of the successive quotients: $h = 75 \cdot (0.52)^n$.
- Use graphing technology to generate a regression equation: $h = 76.608 \cdot (0.516)^n$.

Select students who use these strategies to share during the discussion. Encourage those who do not think an exponential model is appropriate to look for an exponential model that best fits the given data.

***Advancing Student Thinking***: Students may not be comfortable with the data not fitting an exponential function exactly. Remind them that real-world data are messy so, when modeling, we must do our best to approximate the data. If an exponential model does a good job at approximating the data set and showing its general trend, then this is a reasonable model to use even though it does not accurately predict or match all of the data.

Consider allowing students to bounce a tennis ball next to a meter stick to notice how the bounce corresponds to the measurements in the table. Manipulatives can help students understand what the variables in the equation represent and will be a good model if students do Activity 3.

## Student Task Statement

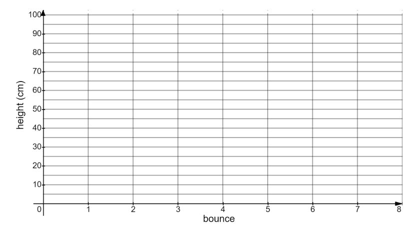
Here are measurements for the maximum height of a tennis ball after bouncing several times on a hard surface.

| $n$, bounce number | $h$, height (centimeters) |
|---|---|
| 0 | 75 |
| 1 | 40 |
| 2 | 21.5 |
| 3 | 10 |
| 4 | 5.5 |

1. Which is more appropriate for modeling the maximum height $h$, in centimeters, of the tennis ball after $n$ bounces: a linear function or an exponential function? Use data from the table to support your answer.
2. Regulations say that a tennis ball dropped on a hard surface should rebound to a height between 53% and 58% of the height from which it is dropped. Does the tennis ball here meet this requirement? Explain your reasoning.
3. 
    a. Remember that coordinate axes are usually called the $x$- and $y$-axes. We usually label coordinates as $(x, y)$. The variables used for our data are labeled $n$ and $h$. Which axes do the $n$ and the $h$ represent?

    b. Graph the data on the coordinate grid.

    c. Describe the shape and relationship of the data.



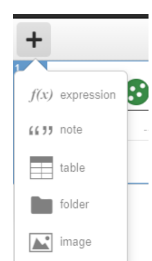

4. Now you are going to graph the data in Desmos and calculate the exponential regression for the model. Follow the instructions in the box below to complete this step.

    a. Access desmos.com/calculator.

    b. Enter your data into a table. To access a table, click on the plus sign at the top left, and select the table option.

    c. Select an appropriate window for your data.

    d. In your second line, type in $y_1 \sim ab^{x_1}$. Select "Log Mode" when the option appears.




5. Write the exponential regression equation, given in Desmos after completing the boxed instructions above, that models the bounce height, $h$, after $n$ bounces for this tennis ball.
6. About how many bounces will it take before the rebound height of the tennis ball is less than 1 centimeter? Explain your reasoning.

## Step 2

- Ask students to share how they decided whether a linear or exponential function is more appropriate for modeling the data. Once the class concludes that a linear model is less suitable, select previously identified students to share how they determined a decay factor to use for their models and the resulting equations. Suggest that students refer to this number as the "rebound factor." Sequence the strategies being presented as shown in the Monitoring Tip.

- Connect the discussion about rebound factors to the data and what those factors tell us about whether the tennis ball satisfies the bounce regulations. Ask questions such as:
  - "Do most of the data support the conclusion?"
  - "How can we explain the third bounce?" (E.g., it is too low, but there could have been a mis-measurement, or the tennis ball could have hit something on the floor.)
- Students may question why the initial value for the regression equation isn't 75 cm. Remind students that a regression equation is a model that minimizes the amount of error between the actual *y*-value and the predicted *y*-value. Consider displaying the residual plot to emphasize this point.

**RESPONSIVE STRATEGY**

Use this routine to prepare students for the whole-class discussion. At the appropriate time, ask students to prepare a visual display that shows their mathematical thinking and reasoning for the first question. Look for two or three visual displays that are different and ask students, "What is the same and what is different?" Then ask students where, in each display, they can "see" the decay factor (for an exponential model) or a constant rate of change (for a linear model). Then ask students to identify where, in each display, they can see when the height of the ball is less than 1 cm.

Compare and Connect (MLR7)

- To follow up on the last question, consider discussing the practical domain in this context. Ask, for instance: "How long could we expect this behavior to continue? Can it go on indefinitely? What is a reasonable domain for our model?"

| DO THE MATH | PLANNING NOTES |
|---|---|
| | |

## Activity 2: Beholding More Bounces (*10 minutes*)

| | |
|---|---|
| **Instructional Routine:** Discussion Supports (MLR8) - Responsive Strategy | |
| **Addressing:** NC.M1.F-IF.5; NC.M1.F-LE.5 | |

This activity continues to examine exponential decay in the context of successive ball bounces. Students use the given data to calculate a rebound factor and use it to write a function that models the relationship between number of bounces and bounce heights. They then use the function to answer questions about the ball and its bounces.

Students also think about the domain of the function and address the fact that this is a discrete context (i.e., it does not make sense to examine $\frac{2}{3}$ of a bounce, so the graphs should not be continuous).

**RESPONSIVE STRATEGY**

Activate or supply background knowledge about discrete and continuous contexts. The number of bounces is counted every time the ball hits the floor. If a student asks about 112 bounces, clarify that would be measuring the time since the ball had been released, not the number of bounces; a "bounce" is the event of hitting the floor and changing direction.

Supports accessibility for: Memory; Conceptual processing

### Step 1

- Invite students to work in pairs or groups, or use visibly random grouping.
- Prompt students to create a scatter plot and calculate the exponential regression to help them approach the task.

***Advancing Student Thinking***: If students struggle to make sense of the graph, remind them that the horizontal axis depicts the number of bounces, not the height of the ball or time. Although height and time are continuous, the number of bounces is discrete.

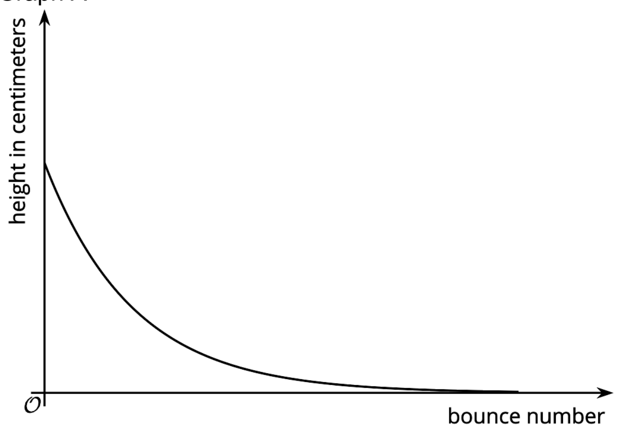
### Student Task Statement

The table shows some heights of a ball after a certain number of bounces.

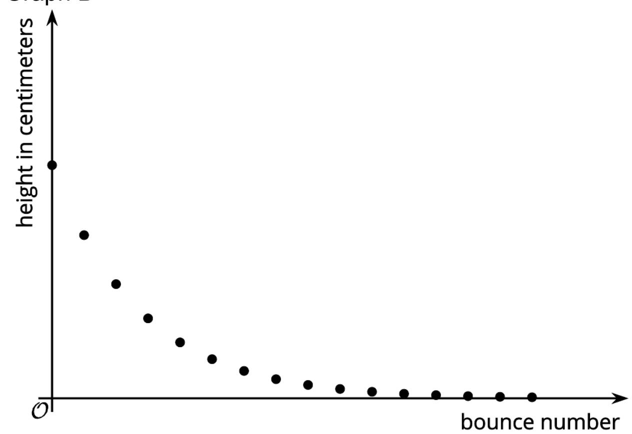
| Bounce number | Height in centimeters |
|---|---|
| 0 | |
| 1 | |
| 2 | 73.5 |
| 3 | 51.5 |
| 4 | 36 |

1. Is this ball more or less bouncy than the tennis ball in the earlier task? Explain or show your reasoning.
2. From what height was the ball dropped? Explain or show your reasoning.
3. Use technology to determine an exponential regression equation that represents the bounce height of the ball, $h$, in centimeters after $n$ bounces.
4. Which graph would more appropriately represent the equation for $h$: graph A or graph B? Explain your reasoning.

Graph A




Graph B




5. Will the $n$-th bounce of this ball ever be lower than the $n$-th bounce of the tennis ball? Explain your reasoning.

### Step 2

Invite students to share how they decided on the bounciness of the balls and to reflect on their reasoning process:

- "How are the data here different from the data in the first activity?" (In this case, the successive quotients are very close to the same value, making the choice of rebound factor more straightforward.)
- "When working with the table, how is calculating the missing value in the row above a given value different from calculating a missing value in the row below?" (To find the missing value in the row above, we need to divide by the rebound factor rather than multiply.)
- "In the final question, why are both the initial height and the bounciness important?" (The value of $h$ is found by multiplying the initial height by the rebound factor $n$ times, so both values matter.)

**RESPONSIVE STRATEGY**

Use this routine to help students reflect on their reasoning process in deciding on the bounciness of the balls. Before the whole-class discussion, arrange students in pairs. Provide the following sentence frames as partners share their ideas: "In the table, I noticed that....", "I interpreted the successive quotients to mean...." or "The equation helped determine the bounciness by...." This will help students make sense of exponential decay through discussing the ball-bouncing situation with a partner.

Discussion Supports (MLR8)

DO THE MATH

PLANNING NOTES

## Activity 3: Which is the Bounciest of All? (*Optional, 40 minutes*)

| | |
|---|---|
| **Instructional Routines:** Aspects of Mathematical Modeling; Stronger and Clearer Each Time (MLR1) - Responsive Strategy | |
| **Building On:** NC.M1.S-ID.6 | **Addressing:** NC.M1.F-LE.5; NC.M1.S-ID.6c |

This activity, designed for an extra class period, engages students in *Aspects of Mathematical Modeling* as they gather and analyze data for bounce heights for multiple balls. Each ball should be sufficiently bouncy to allow measurement of at least four bounces. Good examples include: tennis balls, basketballs, super balls, golf balls, and soccer balls. It will also be important to find a surface that is hard, flat, and level. Any padding will dampen the bounces, and any slant or irregularity on the surface will affect the direction of the bounce. A tiled or concrete floor, or a flat and paved surface outdoors should work. Students will need measuring tapes and may need some practice gathering the data.

Notice how students record the bounce heights. Recording these heights to the nearest inch or centimeter will already be challenging and anything beyond that is too much precision. This activity is a good opportunity to choose a degree of precision appropriate to the context and the measuring device used (MP6).

Making graphing or spreadsheet technology available gives students an opportunity to choose appropriate tools strategically (MP5).

### Step 1

- Ask students to arrange themselves in small groups or use visibly random grouping. Give each group a measuring tape and a ball. Explain to students that their job is to determine the rebound factors of several balls by gathering data on their rebound heights. They then need to use mathematics to model the relationship between the number of bounces and the height of a bounced ball.
- Students should drop them all from the same height. Consider letting students realize this on their own.

RESPONSIVE STRATEGIES

Leverage choice around perceived challenge. Invite students to select one or two balls to bounce, instead of three. Chunking this task into more manageable parts may also benefit students who benefit from additional processing time.

Supports accessibility for: Organization; Attention; Social-emotional skills

**Monitoring Tip:** Monitor for how students process their data and decide on an appropriate factor to quantify the bounciness of each ball. Students should now be comfortable with the fact that the data is not exactly exponential but may still choose different ways for deciding on an appropriate exponential decay factor.

Here is a typical rebound factor for several types of balls:

- Tennis ball: 0.5 to 0.6
- Super ball: 0.9
- Basketball: about 0.6

***Advancing Student Thinking***: Students may struggle to measure the heights of the bounces. Consider allowing phones or other technology that can record a video of the bounces so that it can be replayed in slow motion.

### Student Task Statement

Your teacher will give your group three different kinds of balls.

Your goal is to measure the rebound heights, model the relationship between the number of bounces and the heights, and compare the bounciness of the balls.

1. Complete the table. Make sure to note which ball goes with which column.

| $n$, number of bounces | $a$, height for ball 1 (cm) | $b$, height for ball 2 (cm) | $c$, height for ball 3 (cm) |
|---|---|---|---|
| 0 | | | |
| 1 | | | |
| 2 | | | |
| 3 | | | |
| 4 | | | |

2. Which one appears to be the bounciest? Which one appears to be the least bouncy? Explain your reasoning.
3. For each one, write an equation expressing the bounce height in terms of the bounce number, $n$.
4. Explain how the equations could tell us which one is the most bouncy.
5. If the bounciest one were dropped from a height of 300 cm, what equation would model its bounce height, $h$?

### Step 2

Depending on available time, you may choose to have groups of students prepare a presentation for sharing their findings or simply discuss the data and findings as a whole class.

- As in the previous task, highlight different methods for estimating the rebound factor (taking the quotient of two successive values, taking an average of successive quotients, or making a general estimate of successive quotients). Also highlight the inherent inaccuracy of bounce height measurements which in turn influence how accurately we should report the successive quotients (MP6). Probably no more than one significant decimal digit should be used.
- Focus the discussion on the meaning of the rebound factors. Ask questions such as:
  - "Does a larger factor mean that the ball is more bouncy or less bouncy?"
  - "Does a larger factor mean that the heights are decreasing more quickly or more slowly?"

**RESPONSIVE STRATEGY**

Use this routine to help students improve their writing by providing them with multiple opportunities to clarify their explanations through two paired conversations. Ask students to take turns sharing the response to the question, "Explain how the equations could tell us which one is the most bouncy." Provide listeners with prompts for feedback that will help their partner add detail to strengthen and clarify their ideas. For example, students can ask their partner, "How did you use the data in the table?" or "Can you explain the rebound factor in your equations?" After 2–3 minutes, arrange students in new pairings, and ask them to share and give feedback to their new partner Next, provide students with 3–4 minutes to revise their initial draft based on feedback from their group. This will help students produce a written generalization for how they can estimate the rebound factor and use successive quotients in a real-world exponential situation.

Stronger and Clearer Each Time (MLR1)

| DO THE MATH | PLANNING NOTES |
| --- | --- |
| | |

## Lesson Debrief (*5 minutes*)

The purpose of this lesson is to have students engage in the process of modeling with exponential functions. They deal with data that do not fit the model perfectly, and among other techniques, learn to use exponential regression equations to create models.

Choose what questions to focus the discussion on, whether students should first have an opportunity to reflect in their workbooks or talk through these with a partner, and what questions will be prioritized in the full class discussion.

Invite students to reflect on the process of modeling with the “messy” data they encountered in this lesson.

- “Do you think the data were completely accurate? Why or why not?” (Depending on the tools used, it can be very hard to get exact measurements of rebound heights, and it gets harder the lower the bounces are. Measurement errors are likely.)
- “What was the purpose of calculating an exponential regression?” (The exponential regression calculates a model that we can use to predict the relationship between the bounce number and the bounce height.)
- “We saw different rebound factors for different pairs of successive data points. How do we decide if an exponential model is still appropriate?” (We might want to see if the differences are small enough to treat them as measurement error. In the lesson, they are all within about 0.05 of 0.5, and most of them are close to about 0.54.)
- “Given the inconsistency, how do we find an appropriate factor to use for our model?” (We might disregard the factor(s) that are very different from the rest, or consider finding an average of the factors.)
- “What level of accuracy should we consider? For example, is 0.5 or 0.54 more appropriate for the rebound factor?” (The rebound factor is more appropriate when rounded to the hundredths place because it represents a percent that the next bounce height is compared to the previous bounce height.)

If time permits, discuss possible limitations of our models. Ask questions such as:

- “Do the models we produce work well after 45 bounces? After 78 bounces? When the rebound height is less than 1 cm?” (Probably not for 45 or 78 bounces because the ball will have stopped bouncing by then. When the rebound height is less than 1 cm, the bottom of the ball may remain in contact with the ground during the “bounce,” which is a different physical situation.)
- “Can we rely on the models to be appropriate if we bounce the ball on a different hard surface?” (No, a different surface may have a different amount of “give.”)

PLANNING NOTES

## Student Lesson Summary and Glossary

Sometimes data suggest an exponential relationship. For example, this table shows the bounce heights of a certain ball. We can see that the height decreases with each bounce.

To find out what fraction of the height remains after each bounce, we can divide two consecutive values: $\frac{61}{95}$ is about $0.642$, $\frac{39}{61}$ is about $0.639$, and $\frac{26}{39}$ is about $0.667$.

All of these quotients are close to $\frac{2}{3}$. This suggests that there is an exponential relationship between the number of bounces and the height of the bounce, and that the height is decreasing with a factor of about $\frac{2}{3}$ for each successive bounce.

| Bounce number | Bounce height in centimeters |
|---|---|
| 1 | 95 |
| 2 | 61 |
| 3 | 39 |
| 4 | 26 |

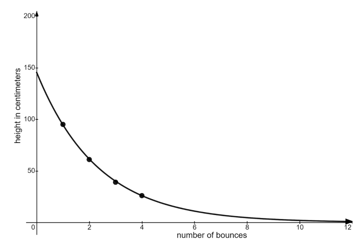
Note that $\frac{2}{3}$ is an estimate of the rebound factor. In fact, the true factor may be 0.65, 0.6672, or something else. It's also possible that the relationship is very close to exponential but not perfect. For this reason, when measurement error is a possibility we do not choose to use very specific decay factors like 0.6672, since we are estimating anyway.

Using technology, such as Desmos, is an efficient way to determine a model that closely aligns to the provided measurements. In order to perform exponential regression, input the data provided in the table into desmos. Then type y1~ab^x1 and select "Log Mode." Desmos will then give you the values of a and b to use for an exponential function that best models the given measurements.

Now we are ready to write an equation that models the height, $h$, of the ball, in cm, after $n$ bounces:

$$h = 145.5 \cdot (0.65)^n$$

Here is a graph of the equation:




This graph shows both the points from the data and the points generated by the equation, which can give us new insights.

## Cool-down: Drop Height (*5 minutes*)

**Addressing:** NC.M1.S-ID.6c; NC.M1.F-LE.5

**Cool-down Guidance:** More Chances
Students will have more opportunities to understand the mathematical ideas in this cool-down, so there is no need to slow down or add additional work to the next lessons. Instead, use the results of this cool-down to provide guidance for what to look for and emphasize over the next several lessons to support students in advancing their current understanding.

### Cool-down

A ball is dropped from a certain height. The table shows the rebound heights of the ball after a series of bounces.

| Bounce number | Height in centimeters |
|---|---|
| 1 | 30 |
| 2 | 6 |
| 3 | 1 |
| 4 | 0 |

Use technology to create a scatter plot and model the data with an exponential regression equation.

From what height, approximately, do you think the ball was dropped? Explain your reasoning.

**Student Reflection:**

In today's lesson you had options of tools to use to work through problems. What helped you most? What helped you least?

## DO THE MATH

## INDIVIDUAL STUDENT DATA

## SUMMARY DATA

## NEXT STEPS

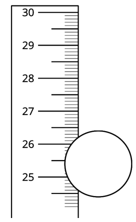
## TEACHER REFLECTION

**What went well in the lesson? What would you do differently next time? What happened today that will influence the planning of future lessons?**

**What do you love most about math? How are you sharing that joy with your students and encouraging them to think about what they love about math?**

## Practice Problems

1. Here is an image showing the highest point of the path of a ball after one bounce.


   Someone is collecting data to model the bounce height of this ball after each bounce. Which measurement for the location of the top of the ball would be the **best** one to record?

   a. 26 cm
   b. 26.4 cm
   c. 26.43 cm
   d. 26.431 cm

2. Function $h$ describes the height of a ball, in inches, after $n$ bounces and is defined by the equation $h(n) = 120 \cdot (\frac{4}{5})^n$.

   a. What is $h(3)$? What does it represent in this situation?

   b. Could $h(n)$ be 150? Explain how you know.

   c. Which ball loses its height more quickly, this ball or a tennis ball whose height in inches after $n$ bounces is modeled by the function $f$ where $f(n) = 50 \cdot (\frac{5}{9})^n$?

   d. How many bounces would it take before the ball modeled by function $h$ bounces less than 12 inches from the surface?

3. After its second bounce, a ball reached a height of 80 cm. The rebound factor for the ball was 0.7. From approximately what height, in cm, was the ball dropped?

   a. 34
   b. 49
   c. 115
   d. 163

4. Which equation is most appropriate for modeling this data?

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $y$ | 79 | 101 | 124 | 158 | 195 | 244 |

   a. $y = 64 \cdot (1.25)^x$
   b. $y = 79 \cdot (1.25)^x$
   c. $y = 79 + 1.25x$
   d. $y = 64 + 22x$

5. The table shows the number of employees and number of active customer accounts for some different marketing companies.

| **Number of employees** | 1 | 2 | 3 | 4 | 10 |
|---|---|---|---|---|---|
| **Number of customers** | 4 | 8 | 13 | 17 | 39 |

   Would a linear or exponential model for the relationship between number of employees and number of customers be more appropriate? Explain how you know.

6. A bank account has a balance of 1,000 dollars. It grows by a factor of $1.04$ each year.

   a. Explain why the balance, in dollars, is a function, $f$, of the number of years, $t$, since the account was opened.

   b. Write an equation defining $f$.

   (From Unit 6, Lesson 11)

7. Rewrite each expression with the least number of exponents (all positive).[2]

   a. $\frac{x^5y^{12}z^0}{x^8y^9}$

   b. $\frac{(y^a)^c}{y^b}$

   c. $\frac{r^5s^3}{rs^3}$

   (From Unit 6, Lesson 2)

8. The table shows the number of people, $n$, who went to see a musical on the $d^{th}$ day of April.

| $d$ | $n$ |
|---|---|
| 1 | 1,534 |
| 2 | 2,324 |
| 3 | 2,418 |
| 4 | 2,281 |
| 5 | 2,350 |
| 6 | 2,394 |
| 7 | 1,720 |

   a. What is the average rate of change for the number of people from day 1 to day 7?

   b. Is the average rate of change a good measure for how the number of people changed throughout the week? Explain your reasoning.

   (From Unit 5)

[2] Adapted from Open Up Resources. Access the full curriculum, supporting tools, and educator communities at openupresources.org.

# Lesson 14: Reasoning about Exponential Graphs (Part One)

## PREPARATION

| Lesson Goals | Learning Targets |
|---|---|
| • Describe (orally and in writing) the effect of changing $a$ and $b$ on a graph that represents $f(x) = a \cdot b^x$.<br>• Use equations and graphs to compare exponential functions. | • I can describe the effect of changing $a$ and $b$ on a graph that represents $f(x) = a \cdot b^x$.<br>• I can use equations and graphs to compare exponential functions. |

### Lesson Narrative

In this lesson, students analyze the graph of an exponential function $f$ given by $f(x) = a \cdot b^x$. In particular, they study the effect of $b$ on the shape of the graph. To observe how $b$ impacts the shape of the graph when $b > 1$ and when $0 < b < 1$, they simultaneously examine several functions of this form, all of which have the same value of $a$. This lesson also provides an opportunity for students to use the term **end behavior** when describing characteristics of functions that exhibit exponential decay: the greater the input value, the closer the output becomes to 0.

Narrow contexts are used in these lessons to encourage greater attention to the growth factors of the different functions, rather than on interpreting the quantities in context. Since students are focusing mainly on understanding how varying $b$ impacts the graph, they are looking at the structure of an exponential function and its graph (MP7).

**What math language will you want to support your students with in this lesson? How will you do that?**

### Focus and Coherence

| Building On | Addressing |
|---|---|
| **NC.8.F.2:** Compare properties of two linear functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).<br><br>**NC.8.F.4:** Analyze functions that model linear relationships.<br>• Understand that a linear relationship can be generalized by $y = mx + b$.<br>• Write an equation in slope-intercept form to model a linear relationship by determining the rate of change and | **NC.M1.F-IF.7:** Analyze linear, exponential, and quadratic functions by generating different representations, by hand in simple cases and using technology for more complicated cases, to show key features, including: domain and range; rate of change; intercepts; intervals where the function is increasing, decreasing, positive, or negative; maximums and minimums; and end behavior.<br><br>(*continued*) |

| | |
|---|---|
| the initial value, given at least two $(x, y)$ values or a graph.<br>• Construct a graph of a linear relationship given an equation in slope-intercept form.<br>• Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of the slope and $y$-intercept of its graph or a table of values. | **NC.M1.F-IF.9:** Compare key features of two functions (linear, quadratic, or exponential) each with a different representation (symbolically, graphically, numerically in tables, or by verbal descriptions).<br><br>**NC.M1.F-LE.5:** Interpret the parameters $a$ and $b$ in a linear function $f(x) = ax + b$ or an exponential function $g(x) = ab^x$ in terms of a context. |

## Agenda, Materials, and Preparation

- **Bridge** (*Optional, 5 minutes*)
- **Warm-up** (*5 minutes*)
- **Activity 1** (*15 minutes*)
  - Technology is required for this activity: Acquire devices that can access Desmos (recommended) or other graphing technology. It is ideal if each student has their own device.
- **Activity 2** (*10 minutes*)
- **Lesson Debrief** (*5 minutes*)
- **Cool-down** (*5 minutes*)
  - M1.U6.L14 Cool-down (print 1 copy per student)

## LESSON

### Bridge (*Optional, 5 minutes*)

**Building On:** NC.8.F.2; NC.8.F.4

This bridge provides students an opportunity to describe the effects of the coefficients in a linear function.

#### Student Task Statement

When The Cookie Kit opened their bakery, they used the function $c(x) = 2.50x + 6.50$ to determine the amount to charge for a custom made-to-order cookie order of $x$ dozen of cookies.

However, due to supply chain problems, the bakery was forced to increase the prices. The function $d(x) = 3.25x + 6.50$ is the function the Cookie Kit now uses for custom orders.

1. How do the functions $c$ and $d$ compare?
2. How do the graphs of $c$ and $d$ compare?

| DO THE MATH | PLANNING NOTES |
|---|---|
| | |

## Warm-up: Spending Gift Money (*5 minutes*)

**Addressing:** NC.M1.F-LE.5

The goal of this warm-up is to practice modeling a situation characterized by exponential decay with an equation.

### Step 1

- Ask students to arrange themselves in pairs or use visibly random grouping.
- Provide students a few minutes of quiet work time and then another minute to share their response with their partner.

***Advancing Student Thinking***: Some students may choose “b” because of the words “$\frac{1}{3}$ of what is left” in the prompt. Ask them what fraction of the quantity is left if $\frac{1}{3}$ of it is spent. If students are still not convinced, ask them how much of the $180 is spent in week 1? How much is left? Use your equation to verify that 120 is left.

### Student Task Statement

Jada received a gift of $180. In the first week, she spent a third of the gift money. She continues spending a third of what is left each week thereafter. Which equation best represents the amount of gift money $g$, in dollars, she has after $t$ weeks? Be prepared to explain your reasoning.

a. $g = 180 - \frac{1}{3}t$

b. $g = 180 \cdot (\frac{1}{3})^t$

c. $g = \frac{1}{3} \cdot 180^t$

d. $g = 180 \cdot (\frac{2}{3})^t$

### Step 2

- Ask students for the correct response.
    - One strategy students may use is to evaluate each equation for some value of $t$. For instance, they may choose 1 for the value of $t$, evaluate the expression, and see if the result makes sense for the amount of money after 1 week. Though this is a valid approach, encourage students to think about what is happening in the situation and see if it is correctly reflected in the way each expression is written. (For instance, in “a,” the term $\frac{1}{3}t$ in the expression would mean “a third of the time in weeks” (rather than a third of the gift money), suggesting that this expression doesn’t accurately reflect the situation. Similarly, in “c”, $\frac{1}{3} \cdot 180^t$ would mean $\frac{1}{3}$ being multiplied by 180 repeatedly ($t$ times), which also does not represent the situation.)
- Make sure students understand that if a third of the balance is spent every week, two-thirds of the balance is what remains every week, so the multiplier is $\frac{2}{3}$, and the amount available after $t$ weeks would be $180 \cdot (\frac{2}{3})^t$.
- If time allows, ask students to explain why some choices are not correct.

| DO THE MATH | PLANNING NOTES |
| --- | --- |
| | |

## Activity 1: Equations and Their Graphs (*15 minutes*)

| |
|---|
| **Instructional Routines:** Graph It; Collect and Display (MLR2) - Responsive Strategy |
| **Addressing:** NC.M1.F-IF.7; NC.M1.F-LE.5 |

The goal of this lesson is to examine how the numbers $a$ and $b$ influence the graph representing a function $f$ defined by an equation of the form $f(x) = a \cdot b^x$. This *Graph It* activity works best when each student has access to devices that can run Desmos because students will benefit from seeing the relationship in a dynamic way.

A money context is used in the first question to facilitate access. An investment worth more money is generally desirable so students will be looking for the function which grows most quickly.

### Step 1

- Keep students in pairs from the previous activity and provide access to graphing technology.
- Read the first question and ask students: "Which function do you predict will result in the most money: $f$, $g$, $h$, or $j$?" Give students a moment to discuss their choice and rationale with their partner.
- Ask students to work with their partner to graph the function and analyze the graphs. Have one partner graph the functions in problem 1 and the other partner graph the functions in problem 2. Students should work together to analyze the graphs and address part b of each question.

**RESPONSIVE STRATEGIES**

As students discuss their ideas with a partner, listen for and collect the language students use to identify and describe how changing the values of "a" and "b" will affect the graph. Write the students' words and phrases on a visual display and update it throughout the remainder of the lesson. Remind students to borrow language from the display as needed. This will help students read and use mathematical language during their partner and whole-group discussions

Collect and Display (MLR2)

**Monitoring Tip:** As students work and discuss in groups, notice those who can articulate how changing $a$ and $b$ in an exponential function affects a graph that represents it.

***Advancing Student Thinking***: Once students have used graphing technology to graph four functions on the same set of axes, they may need help choosing a graphing window so that they can see salient features of all four graphs. They may also need help understanding how to determine which graph represents which function. (Some tools make it easier to distinguish than others.)

### Student Task Statement

1. Each of the following functions, $f$, $g$, $h$, and $j$, represents the value of an investment in a different company, in dollars, as a function of time $x$, in years. They are each written in the form $m(x) = a \cdot b^x$.

$f(x) = 50 \cdot 2^x$
$g(x) = 50 \cdot 3^x$
$h(x) = 50 \cdot (\frac{3}{2})^x$
$j(x) = 50 \cdot (0.5)^x$

   a. Use graphing technology to graph each function on the same coordinate plane and check your prediction.

   b. Explain how changing the value of $b$ changes the graph.

2. Here are equations defining functions $p$, $q$, and $r$. They are also written in the form $m(x) = a \cdot b^x$.

$p(x) = 10 \cdot 4^x$
$q(x) = 40 \cdot 4^x$
$r(x) = 100 \cdot 4^x$

   a. With your partner, make a prediction about what will be the same and what will be different about the graphs of the three functions.

   b. Use graphing technology to graph each function and check your prediction.

   c. Explain how changing the value of $a$ changes the graph.

## Are You Ready For More?

Consider stock investments whose balances are given by the following functions:

a. $f(x) = 10 \cdot 3^x$    b. $g(x) = 3^{x+2}$    c. $h(x) = \frac{1}{2} \cdot 3^{x+3}$

Which stock is doing the best? Does your choice depend on $x$?

## Step 2

- Invite previously selected students or groups to share their responses. Discuss:
  - "The rules for all functions have the form $a * b^x$. On the graph, where do you see the value of $a$ for each function? How does the size of $a$ affect the graph?" (The $a$ is the $y$-intercept. The greater the $a$, the higher it is on the vertical axis.)
  - "On the graph, where do you see the value of $b$ for each function? How does the size of $b$ affect the graph?" (The $b$ appears in the steepness of the curve. The larger the $b$, the steeper the curve, or the more quickly the quantity is growing.)
  - "How is the graph of $j$ different than other graphs?" (It represents a situation characterized by exponential decay: $j(x)$ decreases as $x$ increases.)
  - "How is the value of $b$ for function $j$ different than those of the other functions?" (The value of $b$ is less than 1.)
- Display the graphs of $h$ and $j$ and discuss the domain and range of each.
  - "What is the domain of each function?" ($x$ represents the time in years, so the domain for both functions is $x \geq 0$.)
  - "What is the range of each function?" ($h(x)$ and $j(x)$ represent the value of the investment in dollars with an initial investment of $50. $h(x)$ is increasing so the range is $h(x) \geq 50$. $j(x)$ is decreasing so the range is $0 < j(x) \leq 50$.)

| DO THE MATH | PLANNING NOTES |
| --- | --- |
| | |

## Activity 2: Graphs Representing Exponential Decay (*10 minutes*)

| **Instructional Routines:** Graph It; Take Turns; Discussion Supports (MLR8) |
| --- |
| **Addressing:** NC.M1.F-IF.9 |

This *Graph It* activity complements the previous one. For a situation characterized by exponential growth, a larger growth factor, $b$, means faster growth. For a situation characterized by exponential decay, however, a smaller value of $b$ (between 0 and 1) corresponds to faster decay. Understanding how the parameter $b$ influences the graph of an exponential function will prepare students to make sense of and model situations involving repeated percentage change later in the unit.

### Step 1

- Display the following scenario for all students to see:

> Suppose you are presented with four functions, $m$, $n$, $p$, and $q$, that describe the amount of money, in dollars, in a bank account as a function of time $x$, in years. Here are equations defining the functions.
>
> $$m(x) = 200 \cdot (\frac{1}{4})^x$$
>
> $$n(x) = 200 \cdot (\frac{1}{2})^x$$
>
> $$p(x) = 200 \cdot (\frac{3}{4})^x$$
>
> $$q(x) = 200 \cdot (\frac{7}{8})^x$$
>
> If the account is yours (and more money is better), which function would you choose? Why?

- Give students a minute of quiet think time.
- Ask for a student who chose the first option ($m$) to quickly share their reasoning.
- Ask for a student who chose the last option ($q$) to quickly share their reasoning.
- Tell students they will now consider the options graphically before confirming their choice.

**RESPONSIVE STRATEGY**

Differentiate the degree of difficulty by beginning with more accessible values. When presenting the scenario, begin by displaying only m(x) and p(x) for students to consider briefly before revealing the remaining functions. In addition, some students may benefit from revisiting findings from the warm-up, in which after spending 1/3 of the gift money, Jada had 2/3 left.

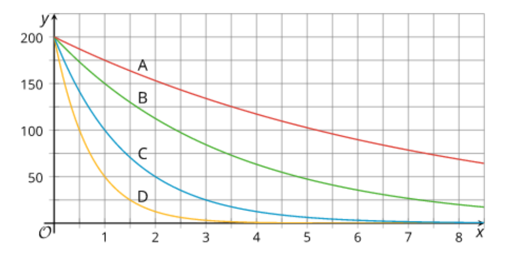
Supports accessibility for: Conceptual processing; Visual-spatial processing

### Step 2

- Keep students in partners from previous activities and provide access to graphing technology to make their matches and answer the questions.
- Have students *Take Turns* with their partner explaining their reasoning for how they matched the equations to the graphs for Question 1. Provide the following sentence frames as *Discussion Supports* for all to see: "The equation _____ matches graph _____ because….", and "I noticed _____ , so I matched…." Encourage students to challenge each other when they disagree. When monitoring the discussions, listen for and amplify key phrases such as "the bases are less than 1," "slowly decaying," or "more vertical curve." This will help students solidify their understanding of how the decay factor affects the graph.

**Student Task Statement**

1. $m(x) = 200 \cdot (\frac{1}{4})^x$
2. $n(x) = 200 \cdot (\frac{1}{2})^x$
3. $p(x) = 200 \cdot (\frac{3}{4})^x$
4. $q(x) = 200 \cdot (\frac{7}{8})^x$




1. Match each equation with a graph. Be prepared to explain your reasoning.
2. Functions $f$ and $g$ are defined by these two equations: $f(x) = 1{,}000 \cdot (\frac{1}{10})^x$ and $g(x) = 1{,}000 \cdot (\frac{9}{10})^x$.
   a. Which function is decaying more quickly? Explain your reasoning.
   b. Use graphing technology to verify your response.

### Step 3

- Facilitate a whole-class discussion focused on the connection between the numbers in the equation (especially the value of $b$) and the features of the graph. Discuss questions such as:
  - "In the first question, what does the point that the graphs share on the $y$-axis say about the situation?" (All of the accounts started with $200.)
  - "How does graph A compare to graph D?" (Graph A appears closer to horizontal, so the function is decaying at a slower rate.)
  - "What does the largest factor ($\frac{7}{8}$) tell us? To which graph does it correspond?" (The function loses only $\frac{1}{8}$ of its value when $x$ increases by 1, so it is decaying the most slowly. Graph A must be the graph of $q(x) = 200 \cdot (\frac{7}{8})^x$ since it shows the slowest decay.)
  - "What might a graph representing a function $v$ given by $v(x) = 200 \cdot (\frac{99}{100})^x$ look like?" (It would look very close to a horizontal line because nearly all of its value remains each time $x$ increases by 1, so it is decaying very slowly.)
  - Will the graph of $v(x) = 200 \cdot (\frac{99}{100})^x$ still approach 0 as $x$ gets larger? (Yes, even though the graph decreases slowly, it is still decreasing and can't go below 0.)

- "What might a graph representing a function $w$ given by $w(x) = 200 \cdot (\frac{1}{100})^x$ look like?" (It will go toward 0 extremely quickly because it loses $\frac{99}{100}$ of its value each time $x$ increases by 1, so it is decaying very quickly.)

- Emphasize that, in situations characterized by exponential growth (when $b > 1$) that fall within the first quadrant, a larger value of $b$ means a curve that is closer to vertical for positive values of $x$. In situations characterized by exponential decay, where $b$ is between 0 and 1, the closer $b$ is to 1, the more the graph approaches a horizontal line. Conversely, the smaller the value of $b$, the more swiftly it heads toward 0 (the more vertical the curve is) before it flattens out and approaches a horizontal line. Tell students that the way a function eventually behaves as its inputs become larger and larger is called its "end behavior." Even though the functions in this activity approach 0 at different rates, they all have the same end behavior.

| DO THE MATH | PLANNING NOTES |
|---|---|
| | |

## Lesson Debrief (*5 minutes*)

The purpose of this lesson is to examine how the parts of a function representing exponential change are reflected in the graph.

Choose whether students should answer the questions by reflecting in their workbooks, talking them through with a partner, or remaining in class discussion mode throughout.

Display the following function and project Desmos for all to see.

$f(x) = a \cdot b^x$

- Ask three students to propose different positive numbers to use in place of *a*. Use the three values to write three functions all with a $b$ value of 2. Do not enter the functions in Desmos yet.
- Ask students to predict how the graphs of the three functions will compare to one another.
  - "What will the three functions have in common?" (They will all have the same steepness; they are all increasing.)
  - "How will they be different?" (The larger the $a$ value, the higher it is on the vertical axes.)
- Graph the three functions using Desmos, adjusting the window if necessary, and ask students for any other observations or questions they may have.
- Ask students, "If $b$ is $\frac{1}{2}$, what would the three graphs look like now?" (The starting positions of the three graphs will be the same as before, and all three graphs will have the same steepness of decay.)
- Change the $b$ value to $\frac{1}{2}$ for each of the Desmos equations, and confirm the students' predictions.
- If time permits, set a fixed value for $a$, ask students to suggest three different positive numbers to use in place of $b$, and repeat.

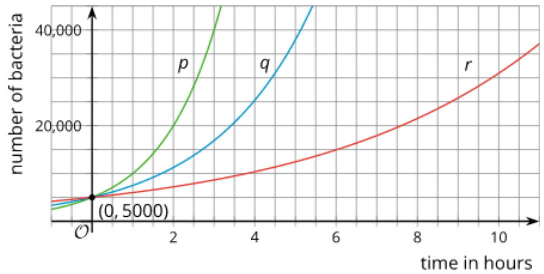
PLANNING NOTES

## Student Lesson Summary and Glossary

An exponential function can give us information about a graph that represents it.

For example, suppose the function $q$ represents a bacteria population $t$ hours after it is first measured and $q(t) = 5,000 \cdot (1.5)^t$. The number 5,000 is the bacteria population measured when $t$ is 0. The number 1.5 indicates that the bacteria population increases by a factor of 1.5 each hour.

A graph can help us see how the starting population (5,000) and growth factor (1.5) influence the population. Suppose functions $p$ and $r$ represent two other bacteria populations and are given by $p(t) = 5,000 \cdot 2^t$ and $r(t) = 5,000 \cdot (1.2)^t$. Here are the graphs of $p$, $q$, and $r$.




All three populations have an initial value of $5,000$, but the graph of $r$ grows more slowly than the graph of $q$ while the graph of $p$ grows more quickly. This makes sense because a population that doubles every hour is growing more quickly than one that increases by a factor of 1.5 each hour, and both grow more quickly than a population that increases by a factor of 1.2 each hour.

The graph also helps us see the **end behavior** of the functions describing the bacteria populations. As time goes on, all of the functions produce larger and larger outputs. They are unlike exponential decay functions, which produce outputs closer and closer to zero as the inputs get larger.

**End behavior of a function**: The characteristics that a function eventually takes on when its input becomes large enough. This might be that the outputs approach a certain value or that they grow without bound. We can also think about end behavior when comparing eventual rates of change of two functions.

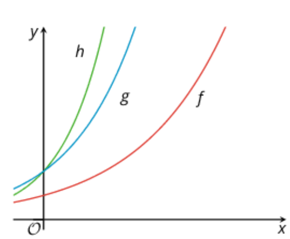
## Cool-down: A Possible Equation (*5 minutes*)

**Addressing:** NC.M1.F-IF.7

**Cool-down Guidance:** Points to Emphasize. The warm-up in Lesson 15 provides an opportunity to discuss the ideas in this cool-down. Use the information you get about student thinking from this cool-down to guide the synthesis of the warm up in Lesson 15.

## Cool-down

Here are three graphs representing three exponential functions, $f$, $g$, and $h$.




The functions $f$ and $h$ are given by $f(x) = 10 \cdot 2^x$ and $h(x) = 20 \cdot 4^x$. Which of the following *could* define the function $g$? Explain your reasoning.

- Equation A: $g(x) = 20 \cdot (1.5)^x$
- Equation B: $g(x) = 20 \cdot (2.5)^x$
- Equation C: $g(x) = 10 \cdot (3.5)^x$
- Equation D: $g(x) = 20 \cdot (4.5)^x$

**Student Reflection:**

True or false: I feel safe to make mistakes and learn from them in math class. Explain why.

## DO THE MATH

## INDIVIDUAL STUDENT DATA

## SUMMARY DATA

## NEXT STEPS

## TEACHER REFLECTION

**What went well in the lesson? What would you do differently next time? What happened today that will influence the planning of future lessons?**

**What evidence have students given that they understand the effects of changing $a$ and $b$ in an exponential function of the form $f(x) = a \cdot b^x$ ? What language do they use or associate with $a$ and $b$ ?**

## Practice Problems

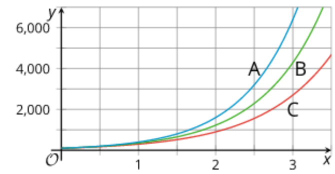
1. Here are equations defining three exponential functions, $f$, $g$, and $h$.

$$f(x) = 100 \cdot 3^x \qquad g(x) = 100 \cdot (3.5)^x \qquad h(x) = 100 \cdot 4^x$$

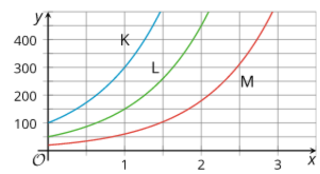
   a. Which of these functions grows the least quickly? Which one grows the most quickly? Explain how you know.

   b. The three given graphs represent $f$, $g$, and $h$. Which graph corresponds to each function?

   c. Why do all three graphs share the same intersection point with the vertical axis?



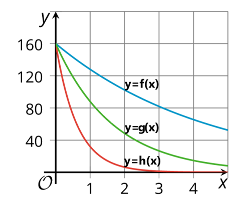

2. Here are graphs of three exponential equations. Match each equation with its graph.

| | | |
|---|---|---|
| a. $y = 20 \cdot 3^x$<br>b. $y = 50 \cdot 3^x$<br>c. $y = 100 \cdot 3^x$ | 1. Graph K<br>2. Graph L<br>3. Graph M |  |



3. The function $f$ is given by $f(x) = 160 \cdot (\frac{4}{5})^x$ and the function $h$ is given by $h(x) = 160 \cdot (\frac{1}{5})^x$.

   If function $g$ is defined by $g(x) = a \cdot b^x$, what can you say about $a$ and $b$? Explain your reasoning.

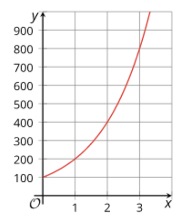



4. Here is a graph of $y = 100 \cdot 2^x$.

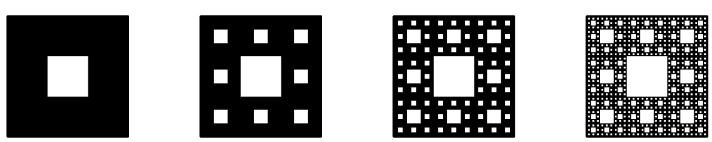
   On the same coordinate plane:

   a. Sketch a graph of $y = 50 \cdot 2^x$ and label it $A$.

   b. Sketch a graph of $y = 200 \cdot 2^x$ and label it $B$.




5. (*Technology required*.) Start with a square with area 1 square unit (not shown). Subdivide it into nine squares of equal area and remove the middle one to get the first figure shown.



   a. What is the area of the first figure shown?

   b. Take the remaining eight squares, subdivide each into nine equal squares, and remove the middle one from each. What is the area of the figure now?

   c. Continue the process and find the area for stages 3 and 4.

   d. Write an equation representing the area $A$ at stage $n$.

   e. Use technology to graph your equation.

   f. Use your graph to find the first stage when the area is less than $\frac{1}{2}$ square unit.

(From Unit 6, Lesson 7)

6.

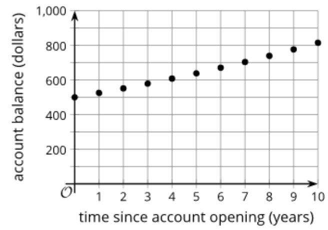
a. Give a positive value of $x$ that would make the inequality $(x^3)^4 > (x^3)^{-4}$ true?

b. Give a positive value of $x$ that would make the inequality $(x^3)^4 < (x^3)^{-4}$ true?

c. Give a positive value of $x$ that would make the equation $(x^3)^4 = (x^3)^{-4}$ true?

(From Unit 6, Lessons 1 and 2)

7. The equation $b = 500 \cdot (1.05)^t$ gives the balance of a bank account $t$ years since the account was opened. The graph shows the annual account balance for 10 years.

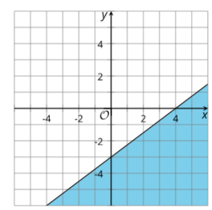



a. What is the average rate of change of the account balance over the 10 years?

b. Is the average rate of change a good measure of how the bank account balance varies? Explain your reasoning.

(From Unit 5)

8. Choose the inequality whose solution region is represented by this graph.




a. $3x - 4y > 12$
b. $3x - 4y \geq 12$
c. $3x - 4y < 12$
d. $3x - 4y \leq 12$

(From Unit 2)

9. During cross country season, Priya and Andre each set aside $\$20.00$ to spend on snacks after practice. Priya buys a Gatorade each day after practice, and Andre buys a granola bar each day after practice. The function $p(x) = -1.25x + 20$ represents the amount of money Priya has left after $x$ days, and the function $a(x) = -0.85x + 20$ represents the amount of money Andre has left after $x$ days.

a. How do the functions $p$ and $a$ compare?

b. How do the graphs of $p$ and $a$ compare?

c. What do $1.25$ and $0.85$ represent in the functions?

(Addressing NC.8.F.2 and NC.8.F.4)

# Lesson 15: Reasoning about Exponential Graphs (Part Two)

## PREPARATION

| Lesson Goals | Learning Targets |
|---|---|
| • Identify the initial value and growth factor of an exponential function given a graph showing two points with non-consecutive input values.<br>• Interpret the intersection of the graphs of two functions that represent a situation. | • When I know two points on a graph of an exponential function, I can write an equation for the function.<br>• I can explain the meaning of the intersection of the graphs of two functions in terms of the situations they represent. |

### Lesson Narrative

This lesson continues to investigate the relationship between the parameters $a$ and $b$ in the expression $a \cdot b^x$ and a graph representing the function $f$ given by $f(x) = a \cdot b^x$. Students start by identifying a function represented by a given graph and using the graph to make sense of a situation. They also examine two abstract graphs, with unlabeled axes, and decide which one represents a given situation. This level of abstraction is appropriate at this stage. It gives students an opportunity to apply what they have learned about the relationship between an exponential context and its graph, and to use graphs to better interpret the contexts. In both cases, they rely on their understanding of the connections between the parameters in an exponential expression and the features of an exponential graph (MP7) to answer questions.

**What strategies or representations do you anticipate students might use in this lesson?**

### Focus and Coherence

| Building On | Addressing |
|---|---|
| **NC.8.F.4:** Analyze functions that model linear relationships.<br>• Understand that a linear relationship can be generalized by $y = mx + b$.Write an equation in slope-intercept form to model a linear relationship by determining the rate of change and the initial value, given at least two $(x, y)$ values or a graph.<br>• Construct a graph of a linear relationship given an equation in slope-intercept form.<br><br>(*continued*) | **NC.M1.A-REI.11:** Build an understanding of why the $x$-coordinates of the points where the graphs of two linear, exponential, and/or quadratic equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$ and approximate solutions using graphing technology or successive approximations with a table of values.<br><br>**NC.M1.F-IF.7:** Analyze linear, exponential, and quadratic functions by generating different representations, by hand in simple cases and using technology for more complicated cases, to show key features, including: domain and range; rate of change; intercepts; intervals where the function is increasing, decreasing, positive, or negative; maximums and minimums; and end behavior. |

| | |
|---|---|
| **NC.8.F.5**: Qualitatively analyze the functional relationship between two quantities.<br>• Analyze a graph determining where the function is increasing or decreasing; linear or non-linear.<br>• Sketch a graph that exhibits the qualitative features of a real-world function. | **NC.M1.F-IF.9:** Compare key features of two functions (linear, quadratic, or exponential) each with a different representation (symbolically, graphically, numerically in tables, or by verbal descriptions).<br><br>**NC.M1.F-BF.1a:** Build linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two ordered pairs (include reading these from a table).<br><br>**NC.M1.F-LE.5:** Interpret the parameters $a$ and $b$ in a linear function $f(x) = ax + b$ or an exponential function $g(x) = ab^x$ in terms of a context. |

## Agenda, Materials, and Preparation

- **Bridge** (*Optional, 5 minutes*)
  - Technology is required for this activity: Acquire devices that can access Desmos (recommended) or other graphing technology. It is ideal if each student has their own device.
- **Warm-up** (*5 minutes*)
- **Activity 1** (*15 minutes*)
- **Activity 2** (*10 minutes*)
- **Lesson Debrief** (*5 minutes*)
- **Cool-down** (*5 minutes*)
  - M1.U6.L15 Cool-down (print 1 copy per student)

## LESSON

### Bridge (*Optional, 5 minutes*)

**Building On:** NC.8.F.4; NC.8.F.5

This bridge gives a function and asks students to describe how the parameters of the function are revealed in the graphical representation. Students need to use technology to graph these functions as the primary focus of this bridge is to connect parameters (of linear and exponential functions) to the graph and context.

#### Student Task Statement

Each function represents the amount in different bank accounts after *t* weeks.

Use technology to create a graph of each function. Describe in words how the money in each account is changing week by week. How do you see this in the graphs?

$$A(t) = 500$$
$$B(t) = 500 + 40t$$
$$C(t) = 500 - 40t$$
$$D(t) = 500(1.5)^t$$
$$E(t) = 500(0.75)^t$$

| DO THE MATH | PLANNING NOTES |
|---|---|
| | |

## Warm-up: Four Functions (*5 minutes*)

| | |
|---|---|
| **Instructional Routine:** Which One Doesn't Belong? | |
| **Building Towards:** NC.M1.F-IF.7 | |

This warm-up encourages students to look closely at functions and articulate the ways in which they are similar or distinct using the *Which One Doesn't Belong?* routine. Given their current work on exponential functions, students might be inclined to wonder if the equations define exponential functions, to look for a growth factor, or to think about the initial value of the function.

### Step 1

- Ask students to arrange themselves in small groups or use visibly random grouping.
- Display the functions for all students to see.
- Give students 1 minute of quiet think time and then time to share their thinking with their small group. In their small groups, ask each student to share their reasoning as to why a particular expression does not belong, and together find at least one reason each item doesn't belong.

### Student Task Statement

Which one doesn't belong? Explain your reasoning.

| | |
|---|---|
| a. $f(n) = 8 \cdot 2^n$ | b. $g(n) = 2 \cdot 8^n$ |
| c. $h(n) = 8 + 2n$ | d. $j(n) = 8 \cdot (\frac{1}{2})^n$ |

### Step 2

- Ask each group to share one reason why a particular function does not belong. Record and display the responses for all to see. After each response, ask the class if they agree or disagree. Since there is no single correct answer to the question of which one doesn't belong, attend to students' explanations and ensure the reasons given are correct.
- During the discussion, highlight ideas that are important to this unit: for example, how the vertical intercept for each function can be identified, that $j$ is a function involving exponential decay, etc. Encourage precision in students' mathematical language: for instance, by reinforcing the use of terms such as "exponential," "growth or decay factor," "initial amount," "base," "exponent," "linear," etc. Do this by prompting for clarification of ideas and revoicing in questions that include academic vocabulary.

**RESPONSIVE STRATEGY**

Provide sentence frames for students to form appropriate responses to discussion questions that incorporate the use of the targeted vocabulary. For example, "Function ____ does not belong because_____." Modeling an example of this framing will help students formulate their answers clearly.

Supports accessibility for: Language; Organization

| **DO THE MATH** | **PLANNING NOTES** |
|---|---|
| | |

## Activity 1: Value of A Computer (*15 minutes*)

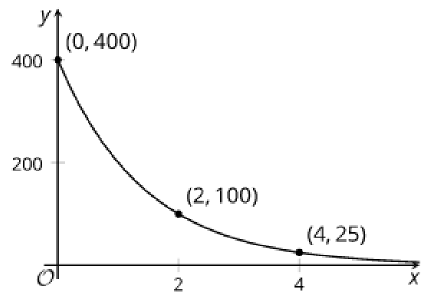
| | |
|---|---|
| **Instructional Routine:** Co-Craft Questions (MLR5) | |
| **Addressing:** NC.M1.F-BF.1a; NC.F-LE.5 | |

In Lesson 14, students used equations defining functions to reason about graphs. Here, they work in the opposite direction, analyzing graphs and creating corresponding equations.

### Step 1

- Have students stay in the same groups from the warm-up.
- Use the *Co-Craft Questions* routine to help students make sense of the situation. Display the task statement and the graph of question 1 (do not display the questions that follow the graph):

Here is a graph representing an exponential function $f$. The function $f$ gives the value of a computer, in dollars, as a function of time, $x$, measured in years since the time of purchase.




- Give students 1 minute to write their own mathematical question about the situation. Have students share their questions with their group members.
- Ask a few groups to share a mathematical question from their group discussion with the class. Listen for and amplify any questions that seek to write an equation, interpret the meaning of various points, or predict future trends.

***Advancing Student Thinking***: If students have trouble getting started, ask them to consider the graph and say what they know is true about the situation that the graph represents. Most likely, they will answer some parts of the first question by doing this, and will be in a better position to figure out any questions they didn't answer.

### Step 2

- Reveal the questions provided in the task and allow students time to work in their group to answer.

***Advancing Student Thinking***: In the first question, the coordinates given are not of two consecutive values of $x$, so students need to reason about the decay factor indirectly. If students are struggling to reason, consider sharing these strategies:

- Try different factors that, when multiplied by $f(0)$ twice, give $f(2)$.
- Use the graph to estimate $f(1)$, and then use that estimate to find a reasonable decay factor.
- Find the factor that takes $f(0)$ to $f(2)$ (or $f(2)$ to $f(4)$), recognize that number as $b^2$, and then find $b$.

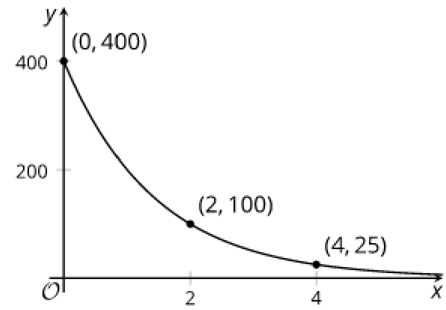
**RESPONSIVE STRATEGY**

Demonstrate and encourage students to use color coding and annotations to highlight connections between representations in a problem. For example, for question 1 use color add labels to the graph to indicate the purchase price of the computer and the value of f when x is 1.

Supports accessibility for: Visual-spatial processing; Conceptual processing

## Student Task Statement

1. Here is a graph representing an exponential function $f$. The function $f$ gives the value of a computer, in dollars, as a function of time, $x$, measured in years since the time of purchase.




   Based on the graph, what can you say about the following?

   a. The purchase price of the computer

   b. The value of $f$ when $x$ is 1

   c. The meaning of $f(1)$

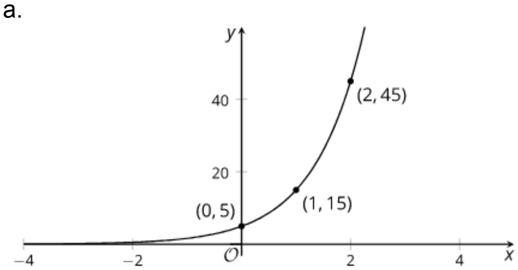
   d. How the value of the computer is changing each year

   e. An equation that defines $f$

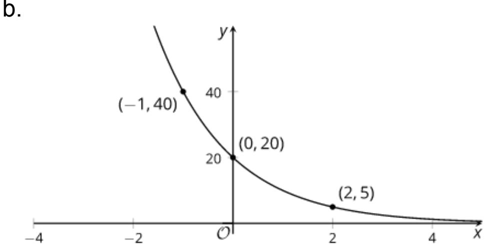
   f. Whether the value of $f$ will reach 0 after 10 years

2. Here are graphs of two exponential functions. For each, write an equation that defines the function and find the value of the function when $x$ is 5.

   a.




   b.




## Are You Ready For More?

Consider a function $f$ defined by $f(x) = a \cdot b^x$.

1. If the graph of $f$ goes through the points $(2, 10)$ and $(8, 30)$, would you expect $f(5)$ to be less than, equal to, or greater than 20?

2. If the graph of $f$ goes through the points $(2, 30)$ and $(8, 10)$, would you expect $f(5)$ to be less than, equal to, or greater than 20?

### Step 3

- Select students to share their responses to the first question. Focus the discussion on how students found the decay factor for the computer. Make sure students understand that the key is to notice that every 2 years, the computer's value is multiplied by $\frac{1}{4}$. This means that the annual decay factor is $\frac{1}{2}$, since $(\frac{1}{2})^2 = \frac{1}{4}$.

- For the second question ask some or all of the following questions to the class:

  – "What information is needed to describe an exponential function?" (the growth or decay factor and the initial value)

- "Where on the graph do we see the initial value?" (the $y$-intercept of the graph)
- "What points on the graph of *a* can we use to help us find the growth factor?" (the points ($(0, 5)$ and $(1, 15)$, or $(1, 15)$ and $(2, 45)$)
- "What points on the graph of $b$ can we use to help us find the decay factor?" (the points $(-1, 40)$ and $(0, 20)$, or $(0, 20)$ and $(2, 5)$)
- "Why might $(-1, 40)$ and $(0, 20)$ be a more strategic choice than $(0, 20)$ and $(2, 5)$ for finding the decay factor of growth?" (The $x$-coordinates of first two points differ by 1, so the quotient of the $y$-coordinates of those points can be used to find the factor of decay. It is a more direct way to find the factor of decay, which is the factor by which the output of $f$ changes when the input $x$ changes by 1. In the second pair, the $x$-coordinate changes by 2.)

| DO THE MATH | PLANNING NOTES |
| --- | --- |
| | |

## Activity 2: Moldy Wall (*10 minutes*)

| |
| --- |
| **Instructional Routines:** Notice and Wonder; Discussion Supports (MLR8) - Responsive Strategy |
| **Addressing:** NC.M1.A-REI.11; NC.M1.F-IF.7; NC.M1.F-IF.9 |

This activity presents students with two graphs without numerical values assigned to any points on the graphs. The only labeled point is where the two graphs intersect. Students begin to make sense of the graph and the context through the *Notice and Wonder* routine. Students will need to use their understanding that tripling each day is a greater growth factor than doubling each day. This is their only means to decide which function represents which situation. Students then interpret the graphs in terms of the situations that they represent.

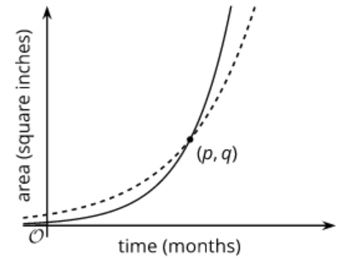
### Step 1

- Display the graph for all to see.
- Give students a moment to observe the graph and ask, "What do you notice? What do you wonder?"
- Invite students to share their observations and questions.
- Provide students time to work on the task in the same groups from the previous activity.

***Advancing Student Thinking***: Some students may think that the dashed graph represents the mold that is tripling because it has a greater vertical intercept. Ask these students to think carefully about the second question.

### Student Task Statement

Here are graphs representing two functions and descriptions of two functions.




- Function $f$: The area of a wall that is covered by mold A, in square inches, doubling every month.
- Function $g$: The area of a wall that is covered by mold B, in square inches, tripling every month.

1. Which graph represents each function? Label the graphs accordingly and explain your reasoning.
2. When the mold was first spotted and measured, was there more of mold A or mold B? Explain how you know.
3. What does the point $(p, q)$ tell us in this situation?

### Step 2

- Facilitate a whole-class discussion by selecting students to share their responses and explanations to the following questions. Focus on how they deciphered the meanings of the graphs.
  - "The solid graph started off lower than the dashed graph. What does that mean in this context?" (The mold population represented by the solid graph is initially smaller or covers a smaller area on the wall than the one represented by the dashed graph.)
  - "After some point, the solid graph passes the dashed one. What does that mean in this context?" (The mold population that started out covering a greater area on the wall is surpassed by the other population, so it is growing at a slower rate.)
  - "What does the intersection of the two graphs mean? What do $p$ and $q$ represent?" (The two mold populations are the same at time $p$ with population covering an area of $q$ square inches.)

**RESPONSIVE STRATEGY**

Use this routine to support whole-class discussion. For each explanation that is shared, ask students to restate what they heard using precise mathematical language. Consider providing students time to restate what they hear to a partner before selecting one or two students to share with the class. Ask the original speaker if their peer was accurately able to restate their thinking. Call students' attention to any words or phrases that helped clarify the original statement. This provides more students with an opportunity to produce language as they interpret the reasoning of others.

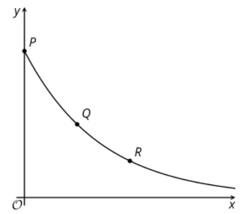
Discussion Supports (MLR8)

| DO THE MATH | PLANNING NOTES |
|---|---|
| | |

## Lesson Debrief (*5 minutes*)

The purpose of this lesson is for students to reason about graphs of exponential functions. They use points on the graph to deduce information about the parameters of the exponential function, and they use both the graph and the equation to answer questions about the situations represented.

Choose what questions to focus the discussion on, whether students should first have an opportunity to reflect in their workbooks or talk through these with a partner, and what questions will be prioritized in the full class discussion.

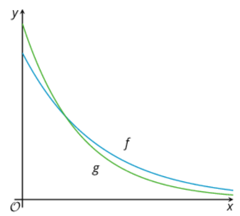



Display the graph of $y = f(x)$ as shown below with points $P$, $Q$ and $R$ labeled.

Tell students that the $x$-coordinate of $Q$ is 1 and the $x$-coordinate of $R$ is 2.

Ask students what information would give us insight into the situation and help us write an equation corresponding to the graph. The following questions may elicit this information if needed.

- "What do you need to know in order to describe how the function is decreasing?" (coordinates of two of the labeled points)
- "If you could find out the coordinates of two of these points, which two would you choose? Why?" ($P$ and $Q$ because the quotient of their $y$-coordinates gives the decay factor and the $y$-coordinate of $P$ gives the initial value.)
- "How would you find the equation using these two coordinates?" (Use the $y$-coordinate of $P$ and the decay factor.)



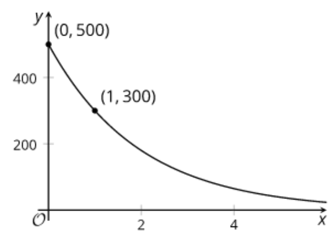

Now, show the graph of $y = f(x)$ along with the graph of $y = g(x)$, another exponential function.

Ask students to compare the two functions by using the following questions:

- "Which function is decreasing faster? How do you know?" ($g$ because it has a larger $y$-intercept than $f$ but then takes smaller values than $f$ as $x$ grows)
- "What does the intersection of the two graphs mean?" (The outputs of $f$ and $g$ are the same for this value of $x$.)

**PLANNING NOTES**

### Student Lesson Summary and Glossary




If we have enough information about a graph representing an exponential function $f$, we can write a corresponding equation. Here is a graph of $y = f(x)$.

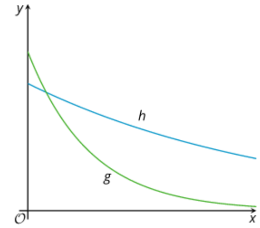
An equation defining an exponential function has the form $f(x) = a \cdot b^x$. The value of $a$ is the starting value or $f(0)$, so it is the $y$-intercept of the graph. We can see that $f(0)$ is $500$ and that the function is decreasing.

The value of $b$ is the growth or decay factor: in this case, decay. It is the number by which we multiply the function's output at $x$ to get the output at $x + 1$. To find the decay factor for $f$, we can calculate $\frac{f(1)}{f(0)}$, which is $\frac{300}{500}$ or $\frac{3}{5}$. So an equation that defines $f$ is:

$$f(x) = 500 \cdot (\tfrac{3}{5})^x$$

We can also use graphs to compare functions. Here are graphs representing two different exponential functions, labeled $g$ and $h$. Each one represents the area of algae (in square meters) in a pond, $x$ days after certain fish were introduced.

- Pond A had 40 square meters of algae. Each day, its area shrinks to $\frac{8}{10}$ of the area on the previous day.
- Pond B had 50 square meters of algae. Each day, its area shrinks to $\frac{2}{5}$ of the area on the previous day.




Can you tell which graph corresponds to which algae population?

We can see that the $y$-intercept of $g$'s graph is greater than the $y$-intercept of $h$'s graph. We can also see that $g$ has a smaller decay factor than $h$ because as $x$ increases by the same amount, $g$ is retaining a smaller fraction of its value compared to $h$. This suggests that $g$ corresponds to pond B and $h$ corresponds to pond A.

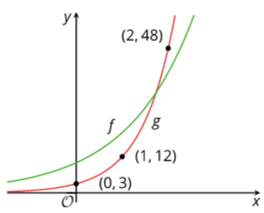
## Cool-down: Two Graphs (*5 minutes*)

**Addressing:** NC.M1.A-REI.11; NC.M1.F-IF.9; NC.M1.F-BF.1a

**Cool-down Guidance:** More Chances
Students will have more opportunities to understand the mathematical ideas in this cool-down, so there is no need to slow down or add additional work to the next lessons. Instead, use the results of this cool-down to provide guidance for what to look for and emphasize over the next several lessons to support students in advancing their current understanding.

### Cool-down

Here are two graphs representing the function $f$ given by $f(x) = 10 \cdot 2^x$ and the function $g$ defined by $g(x) = a \cdot b^x$.




1. Is $b$ greater than or less than 2? Explain how you know.
2. Write an equation that defines $g$. Show your reasoning.
3. $f$ and $g$ represent the number, in thousands, of social media followers of two organizations as a function of years since 2010. What does the intersection of $f$ and $g$ mean in this context?

**Student Reflection:**

Now that you have been analyzing exponential function equations and graphs, what are one or two skills you have gotten really good at doing?

**DO THE MATH**

## INDIVIDUAL STUDENT DATA

## SUMMARY DATA

## NEXT STEPS

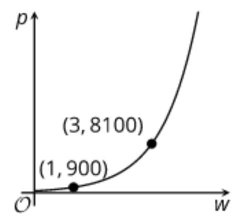
## TEACHER REFLECTION

**What went well in the lesson? What would you do differently next time? What happened today that will influence the planning of future lessons?**

**In Lesson 14, students reasoned from equation to graph while today, they did the opposite and reasoned from graph to equation. What strengths did you see in student reasoning overall? Did they do better with one way? How can you support their reasoning in the other way?**

## Practice Problems

1. Here is a graph of $p$, an insect population, $w$ weeks after it was first measured. The population grows exponentially.

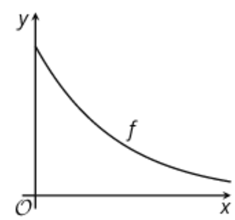



   a. What is the weekly growth factor for the insect population?

   b. What was the population when it was first measured?

   c. Write an equation relating $p$ and $w$.

2. Here is a graph of the function $f$ defined by $f(x) = a \cdot b^x$.




   Select **all** possible values of $b$.

   a. $0$

   b. $\frac{1}{10}$

   c. $\frac{1}{2}$

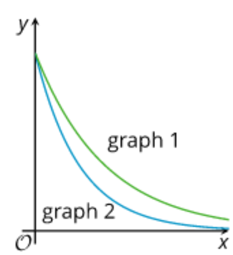
   d. $\frac{9}{10}$

   e. $1$

   f. $1.3$

   g. $\frac{18}{5}$

3. The function $f$ is given by $f(x) = 50 \cdot (\frac{1}{2})^x$, and the function $g$ is given by $g(x) = 50 \cdot (\frac{1}{3})^x$.

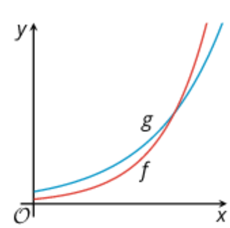



   Here are graphs of $f$ and $g$.

   Kiran says that since $3 > 2$, the graph of $g$ lies above the graph of $f$ so graph 1 is the graph of $g$ and graph 2 is the graph of $f$.

   Do you agree? Explain your reasoning.

4. The function $f$ is defined by $f(x) = 50 \cdot 3^x$. The function $g$ is defined by $g(x) = a \cdot b^x$.




   Here are graphs of $f$ and $g$.

   a. How does $a$ compare to 50? Explain how you know.

   b. How does $b$ compare to 3? Explain how you know.

5. A ball was dropped from a height of 150 cm. The rebound factor of the ball is 0.8. About how high, in centimeters, did the ball go after the third bounce?

   a. 77
   b. 96
   c. 234
   d. 293

   (From Unit 6, Lesson 13)

6. The dollar value of a car is a function, $f$, of the number of years, $t$, since the car was purchased. The function is defined by the equation $f(t) = 12,000 \cdot (\frac{3}{4})^t$.

   a. How much was the car worth when it was purchased? Explain how you know.

   b. What is $f(2)$? What does this tell you about the car?

   c. Sketch a graph of the function $f$.

   d. About when was the car worth \$6,000? Explain how you know.

   (From Unit 6, Lesson 12)

7. (*Technology required*.) The equation $y = 600,000 \cdot (1.055)^t$ represents the population of a country $t$ decades after the year 2000. Use graphing technology to graph the equation. Then, set the graphing window so that you can simultaneously see points on the graph representing the population predicted by the model in 1980 and in the year 2020. What graphing window did you use?

   (From Unit 6, Lesson 7)

8. Rewrite each expression using the fewest number of exponents.

   a. $\frac{3x^{-9}}{6x^{-11}}$ b. $\frac{a^2b^{-3}}{a^2b^{-1}}$ c. $(\frac{2x}{y})^{-5}$ d. $(\frac{c^4d}{c^{-2}})^{-4}$

   (From Unit 6, Lessons 1 and 2)

9. A triathlon athlete runs at an average rate of 8.2 miles per hour, swims at an average rate of 2.4 miles per hour, and bikes at a rate of 16.1 miles per hour. At the end of one training session (during which she did not run), she swam and biked more than 20 miles in total.

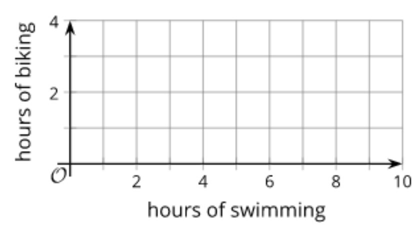
   a. Is it possible that she swam and biked for the following amounts of time in that session? Show your reasoning.

      i. Swam for 0.5 hour and biked 1.25 hours.

      ii. Swam for $\frac{1}{3}$ hour and biked for 70 minutes.

   b. Write an inequality to represent the relationship between the time she swam and biked, in hours, and the total distance she traveled. Be sure to specify what each variable represents.

   c. Use your inequality to graph a solution set that represents all the possible combinations of swimming and running times that meet the distance constraint (regardless of whether the times are realistic).




   (From Unit 3)

10. (*Technology required*.) The population of five different cities after $x$ years is described by the following functions, in thousands of people:

   - City R: $R(x) = 200$
   - City S: $S(x) = 200(1.1)^x$
   - City T: $T(x) = 200 + 1.1x$
   - City U: $U(x) = 200(0.9)^x$
   - City V = $V(x) = 200 - 1.1x$

   Use technology to create a graph of each function.

   Describe in words how the population of each city is changing year by year. How is this represented in the graph?

   (Addressing NC.8.F.4 and NC.8.F.5)

# Lesson 16: Functions Involving Percent Change

## PREPARATION

| Lesson Goals | Learning Targets |
|---|---|
| • Calculate the growth or decay factor given the percent change.<br>• Write an exponential function in the form $f(x) = ab^x$ when given the percent change. | • I can calculate the growth or decay factor given the percent change.<br>• I can write an exponential function to represent a situation where there is repeated percent change. |

### Lesson Narrative

This is the first of two lessons that develop students' ability to represent situations where there is a repeated application of a percent change. Students investigate the effect of interest rates, given as percent increase, that are applied annually and learn that this process is called “compounding.” The same principle applies for repeated percent decrease, which is explored in the second activity involving the income from movie ticket sales.

Students need to recognize that repeatedly applying a percent change such as a 5% increase is the same as repeatedly multiplying by a single factor, in this case, the factor 1.05. This is an example of looking for and expressing regularity in repeated reasoning (MP8).

In these situations, the rate is given as a percent change such as a 5% increase or a 12% decrease. Students use the rates to write the factor $b$ for the exponential function $f(x) = ab^x$. There are different reasoning strategies for writing the factor.

- One strategy is to combine the percentages first and then write the percent into decimal form. For example, given a 5% increase, students may reason, that after each compounding period, there is $100\% + 5\% = 105\%$. This can be written as 1.05 and is the growth factor. Given a 12% decrease, a student using similar reasoning, concludes that $100\% - 12\% = 88\%$. This can be written as 0.88 and is the decay factor.
- The second strategy is to write the percent into decimal form then add or subtract from 1. For example, a 5% increase would first be written as 0.05 and then added to 1: $1 + 0.05 = 1.05$. A 12% decrease would be written as 0.12 and then subtracted from 1: $1 - 0.12 = 0.88$. This is often expressed as $(1 + r)$ or $(1 - r)$ where $r$ is the rate in decimal form.

This lesson emphasizes the first strategy to help students develop an understanding of the growth or decay factors in contextual situations. This will be important when students must interpret growth and decay factors in the next lesson. Being able to contextualize and decontextualize is an example of reasoning abstractly and quantitatively (MP2).

**How is the approach of this lesson similar and different from other ways you have taught these concepts or procedures?**

## Focus and Coherence

| Building On | Addressing | Building Towards |
|---|---|---|
| **NC.7.EE.3:** Solve multi-step real-world and mathematical problems posed with rational numbers in algebraic expressions.<br>• Apply properties of operations to calculate with positive and negative numbers in any form.<br>• Convert between different forms of a number and equivalent forms of the expression as appropriate.<br><br>**NC.7.RP.3:** Use scale factors and unit rates in proportional relationships to solve ratio and percent problems. | **NC.M1.F-BF.1a:** Write a function that describes a relationship between two quantities.<br>**a.** Build linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two ordered pairs (include reading these from a table). | **NC.M1.F-IF.8b:** Use equivalent expressions to reveal and explain different properties of a function.<br>**b.** Interpret and explain growth and decay rates for an exponential function. |

## Agenda, Materials, and Preparation

Desmos or calculators are suggested for this lesson. It is ideal if each student has their own device.

- **Bridge** (*Optional, 5 minutes*)
- **Warm-up** (*5 minutes*)
- **Activity 1** (*15 minutes*)
- **Activity 2** (*10 minutes*)
- **Lesson Debrief** (*5 minutes*)
- **Cool-down** (*5 minutes*)
  - M1.U6.L16 Cool-down (print 1 copy per student)

## LESSON

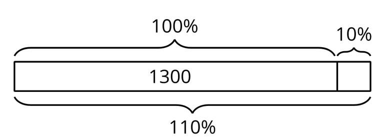
### Bridge (*Optional, 5 minutes*)

**Building On:** NC.7.RP.3

In this lesson and the next, students will need to be able to express a quantity that changes by a given percentage repeatedly. For example, a credit card has a \$100 balance and is charged 12% interest each year. How much do you owe after 5 years? An expression representing the new balance is $100(1.12)^5$. In order for this to make sense, they first need to express the result of a percent increase or decrease using only multiplication. This activity helps to build that capacity.

#### Student Task Statement

An article in the paper says that the local high school's student population will increase by 10% next year. Diego knows that this year, about 1,300 students attend the high school, and he wants to figure out next year's population. First, he draws this diagram.




Diego multiplies 1300 by 1.1 and gets 1430. Explain what he did and why this is correct.

**DO THE MATH**

**PLANNING NOTES**

## Warm-up: Dandy Discounts (*5 minutes*)

**Building On:** NC.7.EE.3

In this warm-up, students explore a situation in which a percent change is applied twice, first on the initial amount and then on the amount that results from the first percent change. The multiplicative way of expressing the change is particularly helpful in such a situation. The example uses a familiar context of sales and discounts. The repeated decay results from multiplying by a factor that is less than 1 more than once.

### Step 1

- Ask students to arrange themselves into pairs or use visibly random grouping.
- Give students 1 minute of quiet think time before asking them to compare their thinking with their partner.

M **Monitoring Tip:** As students work, look for and listen for the following strategies to calculate the price Priya pays:
- using multiplication to calculate 25%, subtracting the result from the original price, and repeating the calculations to apply the second discount
- using multiplication to calculate 75% and repeating the calculation to apply the second discount

Identify students who use each strategy and let them know that they may be asked to share. Include at least one student who does not typically volunteer.

***Advancing Student Thinking***: Some students may need to revisit how to write 25% in decimal form. Remind students that *percent* means *per one hundred* and can be written as $\frac{25}{100} = 0.25$.

### Student Task Statement

All books at a bookstore are 25% off. Priya bought a book originally priced at \$32. The cashier applied the storewide discount and then took another 25% off for a coupon that Priya brought. If there was no sales tax, how much did Priya pay for the book? Show your reasoning.

### Step 2

- Ask selected students to share their responses and briefly describe their calculations. If the second strategy is not used by a student, share with students as another possible strategy. Record the calculations for all to see. For example:

| Strategy 1 | Strategy 2 |
|---|---|
| $32 \cdot 0.25 = 8$<br>$32 - 8 = 24$<br><br>$24 \cdot 0.25 = 6$<br>$24 - 6 = 18$ | $32 \cdot 0.75 = 24$<br>$24 \cdot 0.75 = 18$ |

- Facilitate a whole-class discussion. Here are possible questions:
  - "How are the calculations similar?" (Both repeat the calculations.)
  - "How are they different? (The first strategy multiplies by 0.25 and then subtracts, while the second strategy multiplies by 0.75.)
  - "What does 0.75 represent in this situation?" (A 25% discount means that Priya pays 75% of the current price. Instead of calculating the amount to subtract, multiplying by 0.75 will result in the price to be paid.)
  - "Why is the price Priya paid not 50% of the original price?" (Because the second discount is on the reduced price of \$24 and not on the original price.)

| DO THE MATH | PLANNING NOTES |
| --- | --- |
| | |

## Activity 1: Owing Interests (*15 minutes*)

| **Instructional Routine:** Three Reads (MLR6) - Responsive Strategy | |
| --- | --- |
| **Addressing:** NC.M1.F-BF.1a | **Building Towards:** NC.M1.F-IF.8b |

In this activity, students make sense of repeated percent increase in a borrowing context. An essential point here is that, in each repetition, the value being increased by a percent is not the same as the initial value. Instead, it includes previous increases. They learn that this process is called "compounding."

To express this repetition more generally, it is important for students to represent the percent increase using multiplication. This is helpful when finding values after any number of years and in writing the function. Similar to the warm-up where students discussed the discounted price as 75% of the current price, students need to be able to recognize that at the end of each year, the balance is 118% of the previous year's balance.

**RESPONSIVE STRATEGY**

Use this Three Reads routine to support reading comprehension of this word problem. Display the description of the situation without the questions. Use the first read to orient students to the situation. Ask students to describe what the situation is about without using numbers (A college student wants to take out a loan to buy book). Use the second read to identify quantities and relationships. Ask students what can be counted or measured without focusing on the values. Listen for, and amplify, the important quantities that vary in relation to each other in this situation: the loan amount, number of years, the annual interest rate, and the amount owed. For the third read, display the questions and ask students to brainstorm possible strategies to answer the first two questions. This helps students connect the language in the word problem and the reasoning needed to solve the problem.

Three Reads (MLR6)

### Step 1

- Ask students what they know about how loans work. If students are unfamiliar with the idea of interest on loans, give a brief overview. Explain that a person or a bank may lend money to someone who needs it. In return, the lender would collect interest, which is a percentage of the loan, until the loan is paid. For instance, if Person A decides to borrow $400 from a lender at a 10% interest rate calculated yearly, then after one year, Person A will owe the lender 110%: 100% for the original borrowed amount and 10% for the interest. ($440 \cdot (1.10) = 440$ or $400 + 0.10(400) = 400 + 40 = 440$.) The amount the customer owes is called the balance of the loan.
- Add that typically, people must start making monthly payments on their loans right away, but payments for student loans can be *deferred* - students can wait to make payments until after they graduate. Depending on the type of loan, the interest will still accumulate while the student is in college, so a student will leave college owing more than the original amount of the loan.
- Keep students in pairs and ask them to work on questions 1 and 2 together.

**Monitoring Tip:** As students work, look for and listen for the following strategies to calculate the amount owed:
- using multiplication to calculate 18%, adding the result to how much she owes, and repeating calculations for years 2 and 3
- using multiplication to calculate 118% and repeating the calculation for years 2 and 3

Identify students who use each strategy and let them know that they may be asked to share. Include at least one student who does not typically volunteer.

### Student Task Statement

Tyler has a full tuition scholarship for their first year of college. To purchase books, they obtain a private student loan for $450 that charges 18% interest. The loan is automatically deferred until 6 months after their final college semester, but it accumulates interest on the unpaid balance. Tyler makes no payments during the first year.

1. How much will they owe at the end of one year? Show your reasoning.
2. Assuming Tyler continues to make no payments to the lender, how much will they owe at the end of two years? Three years?

### Step 2

- After the second question, ask selected students to share their responses and briefly describe their calculations. Again, record the calculations for all to see. Use a table like the one shown below.

| Year | Strategy 1 | Strategy 2 | Amount owed |
|---|---|---|---|
| 0 | | | $450$ |
| 1 | $450 \cdot 0.18 = 81$<br>$450 + 81 = 531$ | $450 \cdot 1.18 = 531$ | $531$ |
| 2 | $531 \cdot 0.18 = 95.58$<br>$531 + 95.58 = 626.58$ | $531 \cdot 1.18 = 626.58$ | $626.58$ |
| 3 | $626.58 \cdot 0.18 = 112.78$<br>$626.58 + 112.78 = 739.36$ | $626.58 \cdot 1.18 = 739.36$ | $739.36$ |

- Ask students: “If you were to calculate the amount owed for years 4 through 10, which strategy would you use and why?” (The second strategy is more efficient as it only uses multiplication instead of having to multiply and add.)
- Ask students: “Why doesn’t the amount owed increase by $81 each year?” (Each year, the interest is on the amount owed. In the second year, the 18% interest is on the total amount owed, which is now $531. This is more than 18% of $450.) Explain to students that this process is referred to as “compounding.”
- Tell students that being able to use only multiplication can help in representing exponential situations concisely. If no one has brought it up yet, remind students that they have seen this issue come up with exponential decay.
- Ask students to continue working with their partner on the remaining questions.

***Advancing Student Thinking*:** Some students may have trouble finding the general expression for the amount owed after $x$ years. Suggest they work with expressions that use the original $450 for each year of the loan until a pattern becomes apparent.

### Student Task Statement

3. To find the amount owed at the end of the third year, Tyler wrote:

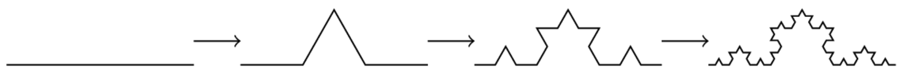
$$450 \cdot (1.18) \cdot (1.18) \cdot (1.18)$$

   Does this expression correctly reflect the amount owed at the end of the third year? Explain or show your reasoning.

4. Write a function rule in the form $f(x) = ab^x$ for the amount Tyler owes at the end of $x$ years without payment.

## Are You Ready For More?

Start with a line segment of length 1 unit. Make a new shape by taking the middle third of the line segment and replacing it by two line segments of the same length to reconnect the two pieces. Repeat this process over and over, replacing the middle third of each of the remaining line segments with two segments each of the same length as the segment they replaced, as shown in the figure.



What is the perimeter of the figure after one iteration of this process (the second shape in the diagram)? After two iterations? After $n$ iterations? Experiment with the value of your expression for large values of $n$.

## Step 3

This discussion should aim to clarify the path toward the function rule $f(x) = 450(1.18)^x$.

- Ask students to share their response to question 3. If needed, illustrate how the amount owed at the end of each year is represented in the calculation

| | |
|---|---|
| $450 \cdot (1.18) \cdot (1.18) \cdot (1.18)$ | 450 is the loan amount |
| $531 \cdot (1.18) \cdot (1.18)$ | 531 is the amount owed after one year |
| $626.58 \cdot (1.18)$ | 626.58 is the amount owed after two years |
| $739.36$ | 739.36 is the amount owed after three years |

- Asks students to share their functions for question 4. Use the following questions:
  - "Why is an exponential function appropriate for this situation?" (The amount owed changes by the same factor every year.)
  - "What does the 450 in the function represent?" (450 is the initial loan amount in dollars.)
  - "How would you explain what 1.18 represents?" (1.18 is the growth factor. It represents 118%, which is how much the amount owed on the loan changes after each year. Because no other payments are made, the initial balance is unchanged. This is 100%. The 18% interest is applied to that initial balance. 118% is written as 1.18 to use as a multiplier.)
  - "What does the $x$ represent?" ($x$ represents the number of years the 18% interest has been applied given that there have been no other changes such as payments or additional loans.)
- Time permitting, ask students how long they think this model accurately will represent the situation. (Tyler will have to start paying back the loan when they graduate, so the balance will go down at that point. Tyler might graduate in 4 or 5 years.)

### RESPONSIVE STRATEGIES

Illustrate the process for generalizing to create an expression or equation. Create a visible display (such as the one included in the Step 3 discussion) that shows how the process of finding the amount owed for any year could be generalized. Use color or annotations to emphasize what changes and what stays the same when calculating the amounts in year 1, 2, and 3. Invite students to describe the meaning of $y=450(1.18)^x$ in their own words.

Supports accessibility for: Visual-spatial processing; Conceptual processing

**DO THE MATH**

**PLANNING NOTES**

## Activity 2: Income from Movie Ticket Sales (*10 minutes*)

| **Instructional Routine:** Discussion Supports - Responsive Strategy (MLR8) | |
|---|---|
| **Addressing:** NC.M1.F-BF.1a | **Building Towards:** NC.M1.F-IF.8b |

In this activity, students make sense of repeated percent decrease to write a function to model the weekly income of a movie since its initial weekend release. An essential understanding is that when given the percent decreases, the percent remaining is used as the decay factor.

### Step 1

- Keep students in pairs.
- Give students a couple of minutes to complete questions 1 and 2 individually.
- Ask students to compare their responses to questions 1 and 2. If there are differences, ask students to take turns explaining their reasoning until they agree.
- Ask students to continue working together on the remaining questions.

***Advancing Student Thinking***: Some students may calculate the weekly income by multiplying by 0.13 and subtracting from the current weekly income. If so, they may have difficulty writing the expression after any number of weeks. Ask students to clarify what the expression represents: (1) the amount of income that is lost each week or (2) the amount of income for that week? Follow up with asking, "Which percentage, 13% or 87%, represents the income for that week?" and "How can that percentage be used in the function?"

### Student Task Statement

The weekend a new movie was released, it had a total income of 270 thousand dollars. The weekly income for each week after initial release decreased by 13%.

1. The first week's income is what percentage of the income the weekend it was released?
2. What is the weekly income one week after initial release? Two weeks? Three weeks?
3. Write a function rule $f(x) =$ ____ for the weekly income after $x$ weeks.
4. Explain what each number in the function rule means in this situation.

### Step 2

The goal of the discussion is to ensure students understand the decay factor is 100% - [percent of decrease], and that it represents the percent remaining.

- Ask students to share their function and explanation for what each number means in the situation. Display the function $f(x) = 270000(0.87)^x$ and label each part as students explain.
- Ask students: "When given a 13% decrease, why multiply by 0.87?" (The function gives the income for the week. If losing 13% each week, then the current week's income is 87% of the previous week's income.)

**RESPONSIVE STRATEGY**

Use this routine to support whole-class discussion. After each student shares, provide the class with the following sentence frames to help them respond: "I agree because…." or "I disagree because…." If necessary, revoice student ideas to demonstrate mathematical language use by restating a statement as a question in order to clarify, apply appropriate language, and involve more students.

Discussion Supports (MLR8)

**DO THE MATH**

**PLANNING NOTES**

## Lesson Debrief (*5 minutes*)

The purpose of this lesson is for students to write exponential functions when given a percent change. Students need to use the percent change to determine the growth or decay factor. The strategy emphasized in this lesson is to reason first with the percent change and then write the percentage in decimal form to use as the factor. Facilitate a discussion using the following questions.

Choose what questions to focus the discussion on, whether students should first have an opportunity to reflect in their workbooks or talk through these with a partner, and what questions will be prioritized in the full class discussion.

- Provide different examples of percent change, such as 45% increase and 0.7% decrease. Ask students to determine the growth or decay factor. ($100\% + 45\% = 145\%$, and the factor would be 1.45; $100\% - 0.7\% = 99.3\%$, and the factor would be 0.993.)
- "Why are exponential functions used to model situations when there is a repeated percent change?" (Each time there is a change, whether it be each year or each hour, the same factor is used as a multiplier.)

**PLANNING NOTES**

### Student Lesson Summary and Glossary

Situations involving a repeated percent change can be represented using exponential expressions.

The following is an example of a repeated percent increase.

When we borrow money from a lender, the lender usually charges *interest*, a percentage of the borrowed amount as payment for allowing us to use the money. The interest is usually calculated at a regular interval of time (monthly, yearly, etc.).

Suppose you received a loan of $500, and the interest rate is 15%, calculated at the end of each year.

- If you make no other purchases or payments, the amount owed after one year is 115% of $500. (100% is the amount of the loan plus the 15% in interest.)
- 115% is written in decimal form and used as the multiplier: $500 \cdot 1.15 = 575$.
- If you continue to make no payments or other purchases, in the second year, the amount owed would be 115% of $575.
- The table shows the calculation of the amount owed for the first 3 years.

| Time in years | Amount owed in dollars |
|---|---|
| 1 | $500 \cdot (1.15)$ |
| 2 | $500 \cdot (1.15)(1.15)$, or $500 \cdot (1.15)^2$ |
| 3 | $500 \cdot (1.15)(1.15)(1.15)$, or $500 \cdot (1.15)^3$ |

- The pattern here continues. Each additional year means multiplication by another factor of $(1.15)$. With no further purchases or payments, after $t$ years, the debt in dollars is given by the expression: $500 \cdot (1.15)^t$

The following is an example of a repeated percent decrease.

A company purchased new equipment at a cost of $100,000. The value of the equipment depreciates (decreases) at a rate of 12% each year.

- The value of the equipment after one year is 88% of $100,000. (100% of the original value minus 12% decrease in value.)
- 88% is written in decimal form and used as the multiplier: $100,000(0.88) = 88,000$.
- After the second year, the value of the equipment has decreased by 12% of the current value of $88,000.
- The table shows the calculation of the value of the equipment for the first 3 years.

| Time in years | Value of equipment |
|---|---|
| 1 | $100,000 \cdot (0.88)$ |
| 2 | $100,000 \cdot (0.88)(0.88)$, or $100,000 \cdot (0.88)^2$ |
| 3 | $100,000 \cdot (0.88)(0.88)(0.88)$, or $100,000 \cdot (0.88)^3$ |

- The pattern here continues. Each additional year means multiplication by another factor of $(0.88)$. After $t$ years, the value of the equipment is given by the expression: $100,000 \cdot (0.88)^t$

## Cool-down: Delayed Payments (*5 minutes*)

**Addressing:** NC.M1.F-BF.1a

**Cool-down Guidance:** More Chances
Students will have more opportunities to understand the mathematical ideas in this cool-down in the next lesson, so there is no need to slow down or add additional work. Instead, use the results of this cool-down to provide guidance for what to look for and emphasize in the next lesson to support students in advancing their current understanding.

### Cool-down

A business owner received a $5,000 loan with 13% interest, charged at the end of each year. Due to a hardship, the business owner was provided an extension for when they needed to begin repaying the loan. Interest was still charged annually.

1. Write an expression to represent the amount owed, in dollars, after the given number of years of making no payments:

   a. after 1 year

   b. after 2 years

   c. after $t$ years

2. Explain why your expression for $t$ years is correct.

**Student Reflection:**

Math concepts build upon each other! What concepts felt familiar to you today?

### DO THE MATH

## INDIVIDUAL STUDENT DATA

## SUMMARY DATA

## NEXT STEPS

## TEACHER REFLECTION

**What went well in the lesson? What would you do differently next time? What happened today that will influence the planning of future lessons?**

**Think about student participation today. Who do you wish you heard or saw more from? Why do you think those students participated less? Reflect on ways you can support students in participating more.**

## Practice Problems

1. In 2011, the population of deer in a forest was 650.
   a. In 2012, the population increases by 15%. Write an expression, using only multiplication, that represents the deer population in 2012.
   b. In 2013, the population increases again by 15%. Write an expression that represents the deer population in 2013.
   c. If the deer population continues to increase by 15% each year, write a function rule $d$ that represents the deer population $t$ years after 2011.
2. Mai and Elena are shopping for back-to-school clothes. They found a skirt that originally cost $30 on a 15% off sale rack. Today, the store is offering an additional 15% off. To find the new price of the skirt, in dollars, Mai says they need to calculate $30 \cdot 0.85 \cdot 0.85$. Elena says they can just multiply $30 \cdot 0.70$.
   a. How much will the skirt cost according to Mai's method?
   b. How much will the skirt cost according to Elena's method?
   c. Explain why the expressions used by Mai and Elena give different prices for the skirt. Which method is correct?
3. Automobiles start losing value, or depreciating, as soon as they leave the car dealership. Five years ago, a family purchased a new car that cost $16,490.

   If the car lost 13% of its value each year, what is the value of the car now?
4. The number of trees in a rainforest decreases each month by 0.5%. The forest currently has 2.5 billion trees.

   Write an expression to represent how many trees will be left in 10 years. Then evaluate the expression.
5. (*Technology required.*) One $1,000 loan charges 5% interest at the end of each year, while a second loan charges 8% interest at the end of each year.
   a. Complete the table with the balances for each loan. Assume that no payments are made and that the interest applies to the entire loan balance, including any previous interest charges.

| $t$, number of years | $b$, loan balance with 5% interest | $c$, loan balance with 8% interest |
|---|---|---|
| 1 | | |
| 2 | | |
| 3 | | |
| $t$ | | |

   b. Which loan balance grows more quickly? How will this be visible in the graphs of the two loan balances, $b$ and $c$, as functions of the number of years, $t$?
   c. Use technology to create graphs representing $b$ and $c$ over time. The graph should show the starting balance of each loan as well as the amount of the loan after 15 years. Write down the graphing window needed to show these points.

6. Lin opened a savings account that pays $5.25\%$ interest annually and deposited $5,000.

   If she makes no deposits and no withdrawals for 3 years, how much money will be in her account?

7. A person loans his friend $500. They agree to an annual interest rate of 5%.

   Write a function rule for computing the amount owed on the loan, in dollars, after $t$ years if no payments are made.

8. The real estate tax rate in 2018 in a small rural county is increasing by $\frac{1}{4}\%$. Last year, a family paid $1,200.

   Which expression represents the real estate tax, in dollars, that the family will pay this year?

   a. $1,200 + 1,200 \cdot (\frac{1}{4})$
   b. $1,200 \cdot (1.25)$
   c. $1,200 \cdot (1.025)$
   d. $1,200 \cdot (1.0025)$

   (From Unit 6, Lessons 9 & 10)

9. Select **all** situations that are accurately described by the expression $15 \cdot 3^5$.

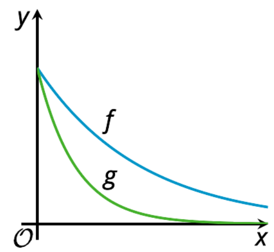
   a. A population of bacteria begins at 15,000. The population triples each hour. How many bacteria are there after 5 hours?
   b. A population of bacteria begins at 15,000. The population triples each hour. How many thousand bacteria are there after 5 hours?
   c. A population of bacteria begins at 15,000. The population quintuples each hour. How many thousand bacteria are there after 3 hours?
   d. A bank account balance is $15. The account balance triples each year. What is the bank account balance, in dollars, after 5 years?
   e. A bank account balance is $15,000. It grows by $3,000 each year. What is the bank account balance, in thousands of dollars, after 5 years?

   (From Unit 6, Lesson 5)

10. Write the following expression with a single positive exponent: $(\frac{1}{2}y^{\frac{1}{3}})^{-4}$.

    (From Unit 6, Lessons 1 and 2)

11. Here are graphs of two exponential functions, $f$ and $g$.

    If $f(x) = 100 \cdot (\frac{2}{3})^x$ and $g(x) = 100 \cdot b^x$, what could be the value of $b$?

    a. $\frac{1}{3}$
    b. $\frac{3}{4}$
    c. $1$
    d. $\frac{3}{2}$

    (From Unit 5)

12. Jada and Priya have $20.00 each to spend at Students' Choice book store, where all students receive a 20% discount. They both want to purchase a copy of the same book, which normally sells for $22.50 plus 10% sales tax.[1]

    - To check if she has enough to purchase the book, Jada takes 20% of $22.50 and subtracts that amount from the normal price. She takes 10% of the discounted selling price and adds it back to find the purchase amount.
    - Priya takes 80% of the normal purchase price and then computes 110% of the reduced price.

    Is Jada correct? Is Priya correct? Do they have enough money to purchase the book?

    (Addressing NC.7.EE.3)

[1] Adapted from https://tasks.illustrativemathematics.org/

# Lesson 17: Interpreting Rates

## PREPARATION

| Lesson Goals | Learning Targets |
|---|---|
| • Approximate solutions to exponential equations in context by graphing.<br>• Interpret and explain growth and decay rates for an exponential function. | • I can solve exponential equations in context by graphing.<br>• I can interpret the growth or decay rate given an exponential function in the form $f(x) = a \cdot b^x$. |

### Lesson Narrative

In the previous lesson, students reasoned with situations that applied repeated percent change such as interest on a loan or caffeine in the body. Given the percent change, students translated the information into a growth or a decay factor and wrote exponential functions.

In this lesson, students write exponential equations involving repeated percent change and use the equations to solve problems. Students may use various strategies for solving. These include using repeated calculations, tables, or graphing technology. Solving exponential equations algebraically is addressed in a later course.

In the second activity, students are given functions and asked to sort them based on whether they represent growth or decay. They interpret the growth or decay factor as a rate given as percent change. For example, the function $f(x) = 500(0.64)^x$ has a decay factor of 0.64. Students may interpret this as “64% of the current value remains.” Next, students consider 100% - ___% = 64% and conclude that there is a 36% decrease.

**Share some ways you see this lesson connecting to previous lessons in this unit. What connections will you want to make explicit?**

## Focus and Coherence

| Building On | Addressing |
|---|---|
| **NC.7.RP.3:** Use scale factors and unit rates in proportional relationships to solve ratio and percent problems.<br><br>**NC.M1.F-IF.7:** Analyze linear, exponential, and quadratic functions by generating different representations, by hand in simple cases and using technology for more complicated cases, to show key features, including: domain and range; rate of change; intercepts; intervals where the function is increasing, decreasing, positive, or negative; maximums and minimums; and end behavior. | **NC.M1.A-CED.1:** Create equations and inequalities in one variable that represent linear, exponential, and quadratic relationships and use them to solve problems.<br><br>**NC.M1.A-REI.11:** Build an understanding of why the $x$-coordinates of the points where the graphs of two linear, exponential, and/or quadratic equations $y=f(x)$ and $y=g(x)$ intersect are the solutions of the equation $f(x)=g(x)$ and approximate solutions using graphing technology or successive approximations with a table of values.<br><br>**NC.M1.F-IF.8b:** Use equivalent expressions to reveal and explain different properties of a function.<br>**b.** Interpret and explain growth and decay rates for an exponential function.<br><br>**NC.M1.F-BF.1a:** Write a function that describes a relationship between two quantities.<br>**a.** Build linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two ordered pairs (include reading these from a table). |

## Agenda, Materials, and Preparation

- **Bridge** (*Optional, 5 minutes*)
- **Warm-up** (*5 minutes*)
  - Graphing technology is optional. Acquire devices that can access Desmos (recommended) or other graphing technology. It is ideal if each student has their own device.
- **Activity 1** (*10 minutes*)
  - Graphing technology is required in this activity.
- **Activity 2** (*15 minutes*)
  - What's the Rate? card sort (print 1 copy for every 2 students and cut up in advance)
- **Lesson Debrief** (*5 minutes*)
- **Cool-down** (*5 minutes*)
  - M1.U6.L17 Cool-down (print 1 copy per student)

## LESSON

### Bridge (*Optional, 5 minutes*)

**Addressing:** NC.7.RP.3

The purpose of this bridge is for students to recognize how to express percent change using only multiplication. For example, "$n$ increased by $9\%$" means there is $109\%$ of $n$ because $100\%+9\%=109\%$. Next, $109\%$ of $n$ would be $\frac{109}{100}\cdot n=1.09n$. In the lesson, students reverse this process by interpreting the factor to describe the percent change. So, given $1.09$, students interpret this as a growth factor meaning there is a $9\%$ increase. This task is aligned to question 8 in Check Your Readiness.

#### Student Task Statement

Express each percent change using an expression that only uses multiplication.

1. $x$ increased by 5%
2. $y$ decreased by 10%
3. $z$ increased by 25%
4. $w$ decreased by 2.5%

| DO THE MATH | PLANNING NOTES |
| --- | --- |
| | |

## Warm-up: Small Town Population (*5 minutes*)

| **Building On:** NC.M1.F-IF.7 | **Addressing:** NC.M1.A-CED.1; NC.M1.F-IF.8b |
| --- | --- |

In this warm-up, students are given a function rule to represent the population growth of a small town. Students interpret the percent increase and use the graph of the function to solve a problem.

### Step 1

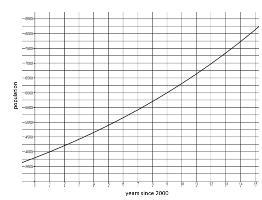
- Ask students to arrange themselves in pairs or use visibly random grouping.
- Provide students with a minute of quiet work time before asking them to share their responses with their partner.

***Advancing Student Thinking*:** Students may confuse 1.052 to mean 52% rather than 5.2%. Ask students how a 52% increase would be written and how a 5% increase would be written.

### Student Task Statement

The population in a small town can be represented by the function $p(t) = 3800(1.052)^t$, where $t$ is the number of years since 2000.

1. What is the percent increase each year? Explain how you know.
2. Use the graph of the equation to find when the population was 7000 people.




### Step 2

- Facilitate a discussion to elicit ideas on interpreting the growth factor as a percent change and using a graph to solve problems.
  - Ask students for their responses and explanations to question 1. As each student explains their reasoning, ask if anyone used a different reasoning. If so, ask them to share.
  - Ask students for their responses to question 2.
    - "How was the graph used to find the solution?" (Identify 7000 on the vertical axis. Move horizontally to the graph of the equation and identify the $x$ coordinate.)
  - Display Desmos for all to see and ask, "How would this be accomplished if using Desmos?"
    - If time allows, ask students to work with their partner to create the graph in Desmos and find the solution before discussing it as a whole class.
    - Use student suggestions to explore making the graph and finding the solution. Include adjusting the axes and using the equation $y = 7000$ to help identify the solution.

**DO THE MATH**

**PLANNING NOTES**

## Activity 1: Cancer Treatment (*10 minutes*)

| **Instructional Routines:** Graph It; Discussion Supports (MLR8) - Responsive Strategy | |
|---|---|
| **Building On:** NC.M1.F-IF.7 | **Addressing:** NC.M1.F-BF.1a; NC.M1.A-CED.1 |

In this *Graph It* activity, students build upon the work in the previous lesson by writing a function for a situation that applies a repeated percent decrease. They use graphing technology to graph the function and use it to solve several problems.

### Step 1

- Keep students in pairs and provide access to Desmos.
- Provide students with a couple of minutes of quiet work time and then another minute to share their responses with their partner.

### Student Task Statement

Iodine-131 is used to treat thyroid cancer. A typical amount used in therapy is 150 millicuries. (A millicurie is a unit used to measure radioactive materials.) Iodine-131 decays at a rate of 8.3% each day.

1. Write a function rule in the form $f(x) = ab^x$ to represent the amount of Iodine-131 in millicuries as a function of the number of weeks $x$.
2. Graph your function using graphing technology and use the graph to answer the following questions. Be prepared to show or explain your reasoning.
    a. After how many days will there be 100 millicuries of iodine-131?
    b. How many days does it take for there to be half of the original amount of iodine-131?
    c. When will there be less than 10 millicuries of iodine-131?

### Are You Ready for More?

A patient received a second iodine-131 treatment 10 days after the first treatment. How many millicuries of iodine-131 would there be after 15 days since the first treatment?

### Step 2

- Ask students to share their function rule and explain what each number in the rule represents. Highlight the reasoning used to write the decay factor as 0.917.
- Ask some students to display their graphs for all to see and share how they used the graph to solve the problems.

**RESPONSIVE STRATEGY**

Provide the following sentence frames for students to use while sharing: "To determine the number of days when ____, I ____" and "The graph helped me see ____."

Discussion Supports (MLR8)

| DO THE MATH | PLANNING NOTES |
|---|---|
| | |

## Activity 2: What's the Rate? (*15 minutes*)

| |
|---|
| **Instructional Routines:** Card Sort; Take Turns; Discussion Supports (MLR8) - Responsive Strategy |
| **Addressing:** NC.M1.F-IF.8b |

In this *Card Sort* activity, students take turns selecting a card, identifying if the factor represents growth or decay, and interpreting what the factor means as a growth or decay rate. This sorting task provides students an opportunity to look for and make use of structure (MP7) and analyze a representation to reason abstractly and quantitatively (MP2).

As students use the *Take Turns* routine to explain their thinking to their partner, encourage them to use precise language and mathematical terms to refine their explanations (MP6).

### Step 1

- Keep students in pairs.
- Tell students they will be sorting cards by identifying the factor of the given function as either growth or decay. They will then interpret what the growth or decay factor means as a rate of change.
- Provide students an example by displaying the following situation for all to see.

> The function rule $f(x) = 1500(1.21)^x$ represents the amount of money owed on a credit card as a function of the number of years.

- Facilitate a whole-class discussion. Use the following questions to model what students will do during the card sort. For each question, provide individual think time before prompting students to discuss with a partner. Ask for student volunteers to share their thinking.
  - "Does the function have a growth or decay factor? Explain how you know." (1.21 is a growth factor because it is greater than 1.)

  - "What does 1.21 mean in this situation?" (The 1.21 means the amount of money owed increases by a rate of 21% each year.) Encourage students to be precise with their interpretations to include the variables from the context and the percent increase or decrease.
- Give one set of pre-cut slips or cards to each pair.
- Ask students to take turns choosing and interpreting the cards.
  - Each person should select a card and read the situation. The student identifies the function as having either a growth factor or a decay factor and interprets the factor as a growth or decay rate.
  - Emphasize that while one partner explains, the other should listen carefully, and they should discuss any disagreements.

**RESPONSIVE STRATEGY**

Provide the following sentence frame to use while sharing: "The ____ factor is ____ and means that the _____ _____ by a rate of ____ each ____." If needed, use student responses to the example from the discussion to model the sentence frame: "The growth factor is 1.21 and means that the amount of money owed increases by a rate of 21% each year."

Discussion Supports (MLR8)

***Advancing Student Thinking***: Students may interpret the decay factor as the percent decrease. For instance, card 4 has a decay factor of 0.77. Students may say that there is a decrease of 77%. Encourage these students to think about whether the outputs of the function represent what is removed or what remains. Ask students, "If we know the percent that remains, how can we determine the percent that was removed?"

### Student Task Statement

Your teacher will give you a set of cards. Each person should select a card, determine whether the given function has a growth factor or a decay factor, and interpret what the growth or decay factor means in the situation. Take turns with your partner sharing your response.

As your partner shares, listen carefully to their interpretation. If you disagree, discuss your thinking and work to reach an agreement. Place the cards into two separate columns, one for growth and one for decay, and then choose another card.

When finished, arrange the column for growth in order from situations involving the least percent change to those with the greatest. Then arrange the column for decay in order from situations involving the least percent change to those with the greatest. List the percent changes in the table below.

| Growth | Decay |
| --- | --- |
| | |

### Step 2

- Ask pairs to volunteer to share their results and explain their interpretations. After a pair shares their interpretation, ask if other pairs agree or disagree. If there is a disagreement, ask pairs to share their thinking and work towards an agreement.
- Attend to the language that students use in their interpretations by giving them opportunities to include information such as specific variables and the percent change.

**DO THE MATH**

**PLANNING NOTES**

## Lesson Debrief (*5 minutes*)

The purpose of this lesson is for students to use exponential functions to solve problems and interpret the growth and decay rates for an exponential function.

Choose what questions to focus the discussion on, whether students should first have an opportunity to reflect in their workbooks or talk through these with a partner, and what questions will be prioritized in the full class discussion.

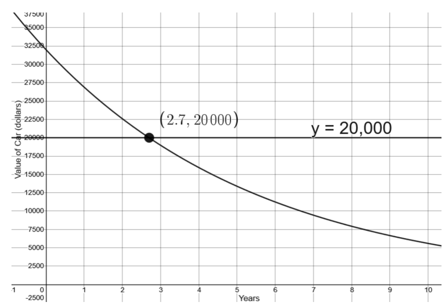
- “In general, how can a graph of an exponential equation help solve problems?” (Points on the graph are solutions. So, using the value you know, you can find the other coordinate. When you have the value of the variable on the vertical axis it helps to graph the horizontal line to find where it intersects with the exponential graph.)
- “How can you determine if the factor is growth or decay?” (If the factor is greater than 1, it is growth; between 0 and 1, it is decay.)
- “How do you interpret a growth factor of 1.82 as a rate?” (1.82 is 182%, which is 100% + 82%, so the rate is an 82% increase.)
- “How do you interpret a decay factor of 0.57 as a rate?” (0.57 is 57% , and 100% - 57% = 43%, so the rate is a 43% decrease.)

**PLANNING NOTES**

### Student Lesson Summary and Glossary

A car was purchased for $\$32,000$. The value of the car depreciates at a rate of $16\%$ each year.

- A $16\%$ decrease means that $84\%$ of the value remains each year ($100\% - 16\% = 84\%$)
- The function rule $v(t) = 32,000(0.84)^t$ represents the value of the car as a function of time, $t$, in years since purchased.




We can use graphing to solve problems such as: “When will the value of the car be $20,000?”

- Graph the function.
- Graph the line $y = 20,000$.
- Identify the point where the horizontal line and the graph of the function intersect: $(2.7, 20,000)$
- The value of the car will be $20,000 approximately 2.7 years after being purchased.

Examining a function rule written in the form $f(x) = ab^x$ allows us to find the percent increase or decrease per unit of time.

The function $f(x) = 300(1.045)^x$ represents the account balance in dollars as a function of the number of years, $x$.

- The growth factor is 1.045, which is 104.5%.
- 100% + 4.5% = 104.5%.
- The account balance increases by a rate of 4.5% each year.

The function $b(t) = 264(0.86)^t$ represents the population of deer as a function of the number of years, $t$.

- The decay factor is 0.86, which is 86%.
- 100% - 14% = 86%.
- The population of deer decreases by 14% each year.

## Cool-down: Interpreting Rates (*5 minutes*)

**Addressing:** NC.M1.F-IF.8b

**Cool-down Guidance:** Points to Emphasize
Spend 5 minutes at the beginning of the next class reviewing the cool-down with students who struggled with this. Have students work on practice problem 6 from Lesson 18 for an additional opportunity to practice.

### Cool-down

For each function rule, identify the percent increase or percent decrease.

1. $f(x) = 135(1.29)^x$
2. $g(x) = 8(0.492)^x$

**Student Reflection:**

The teaching strategies being used to help me learn are:

a. Great! b. Good, but I'd like something different. c. Not helping me at all

Feel free to give details on what teaching strategies are helping most and which you'd change.

### DO THE MATH

### INDIVIDUAL STUDENT DATA

### SUMMARY DATA

## NEXT STEPS

## TEACHER REFLECTION

**What went well in the lesson? What would you do differently next time? What happened today that will influence the planning of future lessons?**

**Reflect on a time your thinking changed about something in class recently. How will you alter your teaching practice to incorporate your new understanding?**

## Practice Problems

1. An algae bloom, if gone untreated, covers a lake at the rate of 2.5% each week. If it currently covers 13 square feet, how many weeks will it take to cover 100 square feet?

2. A computer valued at $1,200 depreciates at a rate of 23% each year. After how long will the computer be approximately valued at $250?

3. Mai used a computer simulation to roll number cubes and count how many rolls it took before all of the cubes came up sixes. Here is a table showing her results.

| $d$, number of cubes | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $r$, number of rolls | 5 | 31 | 143 | 788 |

Would a linear or exponential function be appropriate for modeling the relationship between $d$ and $r$? Explain how you know.

(From Unit 6, Lesson 13)

4. The table shows the height of a ball after different numbers of bounces.

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $h$ | 83 | 61 | 46 | 35 | 26 |

   a. Can the height, $h$, in centimeters, after $n$ bounces be modeled accurately by a linear function? Explain your reasoning.

   b. Can the height, $h$, after $n$ bounces be modeled accurately by an exponential function? Explain your reasoning.

   c. Create a model for the height of the ball after $n$ bounces and plot the predicted values with the data.

   d. Use your model to estimate the height the ball was dropped from.

   e. Use your model to estimate how many bounces it takes before the rebound height is less than 10 cm.

(From Unit 6, Lesson 13)

5. Match each function rule with its corresponding percent change.

| Function | Percent change |
|---|---|
| a. $f(x) = 200(0.75)^x$<br>b. $g(x) = 50(1.25)^x$<br>c. $h(x) = 25(3)^x$<br>d. $j(x) = 75(0.97)^x$ | 1. 3% decrease<br>2. 25% increase<br>3. 25% decrease<br>4. 200% increase |

6.

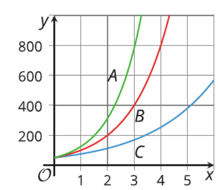
   a. Give a positive value for $x$, such that $x^4 > x^{-4}$.

   b. Give a positive value for $x$, such that $x^4 < x^{-4}$.

   c. Give a positive value for $x$, such that $x^4 = x^{-4}$.

(From Unit 6, Lessons 1 and 2)

7. Here are the graphs of three equations: $y = 50 \cdot (1.5)^x$, $y = 50 \cdot 2^x$, and $y = 50 \cdot (2.5)^x$




Which equation matches each graph? Explain how you know.

(From Unit 5)

8. The bungee jump in Rishikesh, India is 83 meters high. The jumper free falls for 5 seconds to about 30 meters above the river.

   a. Draw a graph of the bungee jump in Rishikesh.

   b. Identify and describe three pieces of important information you can learn from the graph of the bungee jump.

(From Unit 5)

9. A major retailer has a staff of 6,400 employees for the holidays. After the holidays, they will decrease their staff by 30%.

How many employees will they have after the holidays?

(Addressing NC.7.RP.3)

# Lesson 18: Which One Changes Faster?

## PREPARATION

| Lesson Goals | Learning Target |
|---|---|
| • Use graphs and calculations to show that a quantity that increases exponentially will eventually surpass one that increases linearly.<br>• Use tables, calculations, and graphs to compare growth rates of linear and exponential functions. | • I can use tables, calculations, and graphs to compare growth rates of linear and exponential functions and predict how the quantities change eventually. |

### Lesson Narrative

This lesson returns to a theme from the beginning of the unit, revisiting the fact that exponential functions grow more quickly, eventually, than linear functions. The first activity compares simple interest (linear growth) with compound interest (exponential growth). Students examine tables and graphs and see that the exponential function quickly overtakes the linear function. In the second activity, the exponential function is deliberately chosen to grow slowly over a large domain, making it less clear whether or not the exponential function will overtake the linear function.

The second activity provides an opportunity for students to strategically use technology (MP5), whether they make a graph (for which they will need to think carefully about the domain and range) or continue to tabulate explicit values of the two functions (likely with the aid of a calculator for the exponential function). This lesson provides multiple opportunities for students to justify their reasoning and critique the reasoning of others (MP3).

**What ways do you see this lesson connecting to previous lessons in this unit? What connections will you want to make explicit?**

### Focus and Coherence

| Building On | Addressing |
|---|---|
| **NC.8.F.2:** Compare properties of two linear functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).<br><br>(*continued*) | **NC.M1.F-IF.7:** Analyze linear, exponential, and quadratic functions by generating different representations, by hand in simple cases and using technology for more complicated cases, to show key features, including: domain and range; rate of change; intercepts; intervals where the function is increasing, decreasing, positive, or negative; maximums and minimums; and end behavior. |

| | |
|---|---|
| **NC.8.EE.8:** Analyze and solve a system of two linear equations in two variables in slope-intercept form.<br>● Understand that solutions to a system of two linear equations correspond to the points of intersection of their graphs because the point of intersection satisfies both equations simultaneously.<br>● Solve real-world and mathematical problems leading to systems of linear equations by graphing the equations. Solve simple cases by inspection.<br><br>**NC.M1.A-REI.11:** Build an understanding of why the x-coordinates of the points where the graphs of two linear, exponential, or quadratic equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$ and approximate solutions using a graphing technology or successive approximations with a table of values. | **NC.M1.F-BF.1a:** Build linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two ordered pairs (include reading these from a table).<br><br>**NC.M1.F-LE.1:** Identify situations that can be modeled with linear and exponential functions, and justify the most appropriate model for a situation based on the rate of change over equal intervals.<br><br>**NC.M1.F-LE.5:** Interpret the parameters $a$ and $b$ in a linear function $f(x) = ax + b$ or an exponential function $g(x) = ab^x$ in terms of a context. |

## Agenda, Materials, and Preparation

Technology is required for this lesson: Acquire devices that can access Desmos (recommended) or other graphing technology. It is ideal if each student has their own device.

- **Bridge** (*Optional, 5 minutes*)
- **Warm-up** (*5 minutes*)
- **Activity 1** (*15 minutes*)
- **Activity 2** (*10 minutes*)
- **Lesson Debrief** (*5 minutes*)
- **Cool-down** (*5 minutes*)
  - M1.U6.L18 Cool-down (print 1 copy per student)

## LESSON

### Bridge (*Optional, 5 minutes*)

**Building On:** NC.8.F.2; NC.8.EE.8; NC.M1.A-REI.11

This bridge has students compare two linear functions in context in preparation for comparing a linear and exponential function in this lesson. Based on assumptions made of the required domain values, the choice of the "better" function is dependent on the relevant values of the domain, another concept explored in this lesson.

#### Student Task Statement

You are a representative for a cell phone company, and it is your job to promote different texting plans. Functions $a$ and $b$ give the cost of each plan, where $t$ is the number of text messages sent.[1]

Plan A: $a(t) = 29.95 + 0.10t$

Plan B: $b(t) = 49.94 + 0.05t$

1. Which plan is the better deal for someone who usually sends around 100 texts per month?
2. Do you think this plan is best for everyone? Explain your reasoning.

[1] Adapted from https://tasks.illustrativemathematics.org/

**DO THE MATH**

**PLANNING NOTES**

## Warm-up: Graph of Which Function? (*5 minutes*)

**Addressing:** NC.M1.F-IF.7

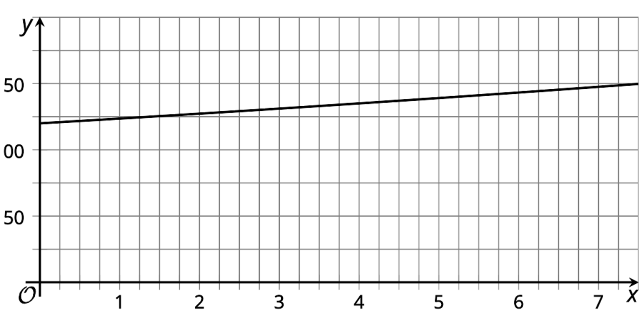
At the beginning of this unit, students compared linear and exponential growth. They return to this comparison in this lesson. This warm-up aims to show that, visually, it could be very difficult to distinguish linear and exponential growth for some domains of the function. While any exponential function eventually grows very quickly, it also can look remarkably linear over a portion of the domain.

**Step 1**

- Ask students to arrange themselves in pairs or use visibly random grouping.
- Provide students a couple of minutes of quiet work time and then another minute to share their responses with their partner.

### Student Task Statement

Here is a graph.

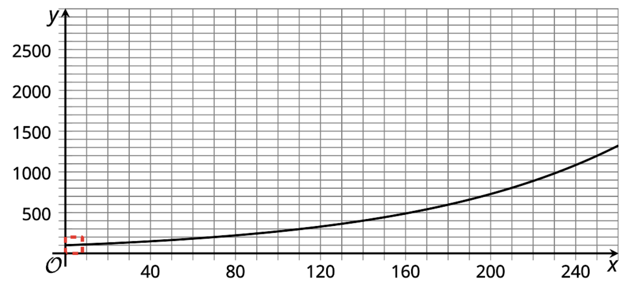



1. Which equation do you think the graph represents, equation "a" or "b"? Use the graph to support and explain your reasoning.
   a. $y = 120 + (3.7) \cdot x$
   b. $y = 120 \cdot (1.03)^x$
2. What information might help you decide more easily whether the graph represents a linear or an exponential function?

### Step 2

- In a whole-class discussion, invite students to share the rationales for their decision and their ideas for improving the clarity of the graph.
- Help students understand that the graph of a linear function always looks like a line regardless of the domain where it is plotted. An exponential function, however, can look linear depending on the domain. Graphs are very helpful for seeing the general behavior of a function but not always for determining what kind of function is being graphed.

- Consider displaying this image of the graph that includes a larger domain as well as the original graphing window (the red rectangle in the lower left). Alternatively, use Desmos to show the graph of $y = 120 \cdot (1.03)^x$ starting with a small window where the graph looks linear and then zooming out until the curve is visible.




## DO THE MATH

## PLANNING NOTES

## Activity 1: Simple and Compound Interests (*15 minutes*)

| | |
|---|---|
| **Instructional Routines:** Graph It; Three Reads (MLR6) | |
| **Addressing:** NC.M1.F.IF.7; NC.M1.F-BF.1a; NC.M1.F-LE.1 | |

In this *Graph It* activity, students compare linear and exponential growth in a context involving simple and compound interest. The initial balances are chosen so that the two options stay pretty close for small values of time. The intersection of their graphs is far enough from 0 that it is not easily noticed with only a few calculations. The graph for simple interest is linear. The graph for compound interest is exponential, but it is relatively flat for small values of time. As the domain values increase, students may notice that the values of the two options get closer and closer, or they may notice from the graph that the gap between the two graphs gets smaller.

### Step 1

- Ask students to arrange themselves in partners or use visibly random grouping.

Use the *Three Reads* routine to support comprehension of this word problem. This routine helps students interpret the language within a given situation needed to create a system of linear inequalities.

- First Read: Without displaying the task, read the context aloud to the class: “A family has $1,000 to invest and is considering two options: investing in government bonds that offer 2% simple interest or investing in a savings account at a bank, which charges a $20 fee to open an account and pays 2% compound interest. Both options pay interest annually.”
  - Ask students: “What is this situation about? What is going on here?” (A family is deciding how to invest money. One account is receiving simple interest; the other is receiving compound interest. One account must pay a fee; the other does not.)
  - Let students know the focus is just on the situation, not on the numbers.
  - Clarify any unfamiliar words (e.g., “simple interest” as compared to “compound interest”).

- Second Read: Display the description and the tables of the situations (but without the problems) and ask a student volunteer to read the problem aloud to the class again.
    - Ask: “What are the quantities in this situation? A quantity is something that can be counted or measured.”
    - Spend less than a minute scribing student responses.
    - Write down all ideas students generate, but listen for and amplify the important quantities that vary (e.g., amount in each account over the years, number of years, interest rates, fee amount).
- Third Read: Invite students to read the situation again to themselves, or ask another student volunteer to read the problem aloud. Then reveal the student task statement, and ask students to read the problem(s) to themselves.
    - Ask students: “How might you approach the problem(s)? What is the first thing you will do?”
    - Spend less than a minute brainstorming possible starting points. Be sure to stop any students who begin to share a complete solution; the goal is to crowdsource starting points.

### Step 2

- Provide students 2–3 minutes of quiet time to begin working on the questions.
- Have students continue working with their groups on the remaining questions.
- Provide access to graphing technology to each group.

M **Monitoring Tip:** Look for students who:
- make different choices for the investment option, as the better option would depend on the length of investment, which is unspecified
- decide to change their earlier choice (particularly after looking at the graph) and can defend the change

### Student Task Statement

A family has $1,000 to invest and is considering two options: investing in government bonds that offer 2% simple interest or investing in a savings account at a bank, which charges a $20 fee to open an account and pays 2% compound interest. Both options pay interest annually.

Here are two tables showing what they would earn in the first couple of years if they do not invest additional amounts or withdraw any money.

Bonds

| Years of investment | Amount in dollars |
|---|---|
| 0 | $1,000 |
| 1 | $1,020 |
| 2 | $1,040 |
| | |
| | |

Savings account

| Years of investment | Amount in dollars |
|---|---|
| 0 | $980 |
| 1 | $999.60 |
| 2 | $1,019.59 |
| | |
| | |

1. Describe the way the investment in bonds grows with simple interest.
2. For the savings account, how are the amounts $999.60 and $1,019.59 calculated?
3. For each option, write an equation to represent the relationship between the amount of money and the number of years of investment.
4. Which investment option should the family choose? Use your equations or calculations to support your answer.
5. Use graphing technology to graph the two investment options and show how the money grows in each.

### Step 3

- Facilitate a whole-class discussion by first surveying the class to see the choices students made.
  - Ask students who selected the bonds as the best investment to explain their reasoning and graphs. Display for all to see.
  - Ask students who selected the savings account to explain their reasoning and graphs. If a group changed their reasoning during group work time, be sure to include their reasoning in the class discussion.
- Use the following discussion questions as needed:
  - "How does the fee for the savings account affect the function?" (It reduces the account balance, at the beginning, by $20.)
  - "Where can we see the fees on the graph?" (The vertical intercept of the graph for the savings account is at 980, which is 20 less than 1,000.)
  - "When might the bonds be the better investment, if ever?" (When the investment is shorter than a decade, as that is when the savings account surpasses it.)
  - "When might the savings account be the better investment, if ever?" (When it is left for more than 10 years, after the compounding effects of interest make up for the initial fees.)
- Point out that the exponential function here (the balance of the savings account) has a relatively slow rate of growth. It takes it a relatively long time before it overtakes the linear function, but it eventually does. When finding which of the two functions eventually produces larger outputs, we are comparing end behavior.

| DO THE MATH | PLANNING NOTES |
|---|---|
| | |

## Activity 2: Reaching 2,000 (*10 minutes*)

| |
|---|
| **Instructional Routines:** Graph It; Notice and Wonder; Compare and Connect (MLR7) |
| **Addressing:** NC.M1.F-IF.7; NC.M1.F-LE.3 |

This *Graph It* activity prompts students to again compare a linear function with an exponential one, but this time without a context, and the exponential function grows much more slowly over a long period of time. Making graphing and spreadsheet technology available gives students an opportunity to choose appropriate tools strategically (MP5).

## Step 1

- Keep students in their groups from the previous activity.
- Present the two equations that define two functions $f$ and $g$: $f(x) = 2x$ and $g(x) = (1.01)^x$.
- Provide 1 minute of quiet time for students to individually *Notice and Wonder* about the two functions.
- Provide 1 minute for group members to share their observations and questions with each other and the class.
- Consider asking the following questions for students to enter the problem:
  - "What is happening in the first function?" (the output increases by 2 every time the input increases by 1)
  - "What is happening in the second function?" (the output is multiplied by 1.01 every time the input increases by 1)
  - "Which function do you think will reach an output of 2000 first?"
- Provide groups 3–4 minutes to finish working on the task in groups.

**Monitoring Tip:** As students are working to decide which function reaches 2,000 first, they may try:
- continuing the table of values with increasingly larger values
- using graphing technology
- finding when $f(x) = 2,000$ and substituting this value ($x = 1,000$) into $g(x)$

Monitor for students who used these contrasting strategies and ask some of these students to prepare to share their thinking with the class. Be mindful to encourage students who may not typically volunteer on their own.

***Advancing Student Thinking***: If students get stuck, suggest they consider increasing by 100 a few times and record the function values in the table.

### Student Task Statement

1. $f(x) = 2x$ and $g(x) = (1.01)^x$
   Complete the table of values for the functions $f$ and $g$.

| $x$ | $f(x)$ | $g(x)$ |
|---|---|---|
| 1 | | |
| 10 | | |
| 50 | | |
| 100 | | |
| 500 | | |
| | | |
| | | |
| | | |

2. Based on the table of values, which function do you think grows faster? Explain your reasoning.
3. Which function do you think will reach a value of 2,000 first? Show your reasoning.

### Are You Ready For More?

Consider the functions $g(x) = x^5$ and $f(x) = 5^x$. While it is true that $f(7) > g(7)$, for example, it is hard to check this using mental math. Find a value of $x$ for which properties of exponents allow you to conclude that $f(x) > g(x)$ without a calculator.

## Step 2

- Facilitate a whole-class discussion about which function will reach 2,000 first.
  - Display a graph of each of the two functions, initially using a window of $-20 \leq x \leq 20$ and $-40 \leq y \leq 40$, which shows $f$ growing faster than $g$. Ask students if this means that the savings bond investment will reach $2000 first.
  - Then zoom out until $y$-values of 2,000 are visible. This will show function $g$ overtaking function $f$. Allow students to confirm or revise their answers. Tell students that when we look at the end behavior of each function, we find that $g$ grows faster than $f$.
  - Use the *Compare and Connect* routine to focus the discussion on the similarities and differences in what the table and graphs showed.
  - Invite previously selected students who used contrasting approaches to share in the order shown in the Monitoring Tip.
  - Ask students, "What is different and what is the same about what you can see in the table and what you can see with the graphs? Which strategy do you believe is the most efficient in determining a solution?"
  - During the discussion, make sure to raise each of these points if not raised by students:
    - The exponential function $g$ grows sufficiently slowly that it looks like it will not catch up with or ever overtake the linear function $f$. This is true for a table of values as well as using a graph.
    - Emphasize that the table needs to be continued for a long time to identify when the values of $g$ become greater than those of $f$. Similarly, the window of the graph needs to be chosen carefully. A very efficient method is to identify that $f(1,000) = 2,000$ and then evaluate $g(1,000)$. Since $g(1,000)$ is much greater than $2,000$, $g$ reaches $2,000$ first. If not already illustrated by students who used graphing, show this dynamic sketch and zoom out. Though for quite a while it doesn't seem like the values of $g$ would ever catch up with those of $f$, if the growth is allowed to take place long enough, exponential eventually surpasses linear.

| DO THE MATH | PLANNING NOTES |
| --- | --- |
| | |

## Lesson Debrief (*5 minutes*)

The purpose of this lesson is to have students compare the patterns of change of linear and exponential functions, using graphs, equations,and tables of values. There is particular emphasis on end behavior of these functions, particularly how exponential functions will eventually outgrow linear functions.

Choose what questions to focus the discussion on, whether students should first have an opportunity to reflect in their workbooks or talk through these with a partner, and what questions will be prioritized in the full class discussion.

Present these pairs of functions and ask the following questions:

- Two linear functions $f$ and $g$: for example, $f(x) = \frac{1}{2}x + 100$ and $g(x) = 2x$
  - "How can we tell which one grows more quickly?" (We find which has the larger slope.)
  - "Will the one that started out with the lower initial value catch up with the function that starts with a higher initial value?" (Yes, this can be seen by graphing or by setting the function rules equal to each other and solving.)
  - "What are some ways to check if one function will overtake another?" (You could use a graph or a table, but these methods will not always work because you may not have looked at large enough $x$-values to tell. A linear function with a larger slope will eventually overtake a linear function with a smaller slope.)
- Two exponential functions $p$ and $q$: for example, $p(x) = 100 \cdot (1.5)^x$ and $q(x) = 5 \cdot 2^x$
  - "How can we tell which one grows more quickly?" (compare the growth factors)
  - "Will the one that started out with the lower initial value catch up with the function that starts with a higher initial value?" (yes)
  - "What are some ways to check if one function will overtake another?" (When comparing exponential growth, a function with a larger growth factor will always overtake a function with a smaller growth factor.)
- A linear function $m$ and an exponential function $n$: for example, $m(x) = 300x + 1,000$ and $n(x) = (0.4) \cdot 2^x$
  - "How can we tell which one grows more quickly?" (Function $m$ increases by addition of $300$, but function $n$ increases by multiplication by $2$. This means that function $n$ is growing faster.)
  - "Will the one that started out with the lower initial value catch up with the function that starts with a higher initial value?" (yes)
  - "What are some ways to check if one function will overtake another? (Look at the type of function: exponential functions will always overtake linear functions in the long run.)

For all of the sets of functions and discussion questions, reiterate that even though an exponential function might seem to be growing too slowly to catch up to a linear function with large numbers for its $y$-intercept and slope, at some value of $x$ down the line, the exponential function will catch up with the linear. This is due to the end behavior of both functions: linear functions have a constant rate of change, but these exponential functions have an increasing rate of change. Remind students how this was the case in the genie activity early in the unit. (The starting value for the linear function was 100,000 times that of the exponential function, and the slope of the linear function was 2,000 while the growth factor of the exponential function was 2. But within 3 weeks, the exponential function surpassed the linear.)

**PLANNING NOTES**

## Student Lesson Summary and Glossary

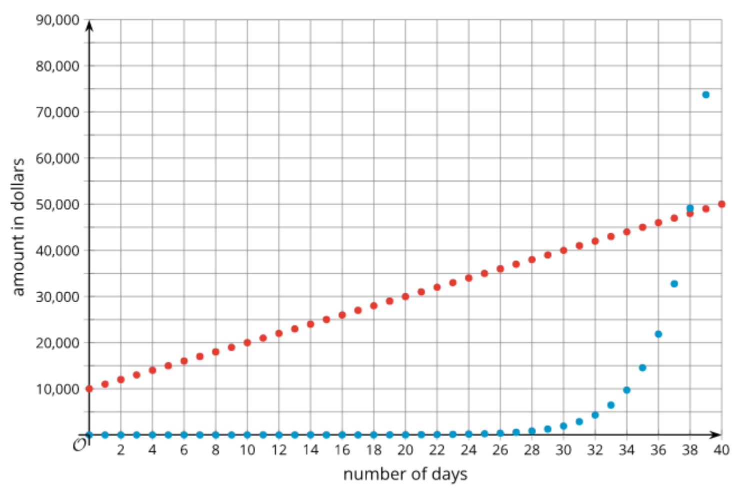
Suppose that you won the top prize from a game show and are given two options. The first option is a cash gift of $\$10,000$ and $\$1,000$ per day for the next 7 days. The second option is a cash gift of 1 cent (or $\$0.01$) that grows tenfold each day for 7 days. Which option would you choose?

In the first option, the amount of money increases by the same amount ($\$1,000$) each day, so we can represent it with a linear function. In the second option, the money grows by multiples of 10, so we can represent it with an exponential function. Let $f$ represent the amount of money $x$ days after winning with the first option and let $g$ represent the amount of money $x$ days after winning with the second option.

| $f$ | $f(x)$ | $g(x)$ |
|---|---|---|
| $x$ | $10,000 + 1,000x$ | $(0.01) \cdot 10^x$ |
| 1 | $11,000$ | $0.1$ |
| 2 | $12,000$ | $1$ |
| 3 | $13,000$ | $10$ |
| ... | ... | ... |
| 6 | $16,000$ | $10,000$ |
| 7 | $17,000$ | $100,000$ |

For the first few days, the second option trails far behind the first. Because of the repeated multiplication by 10, however, after 7 days it surges past the amount in the first option.

What if the growth factor factor of growth is much smaller than 10? Suppose we have a third option, represented by a function $h$. The starting amount is still $\$0.01$ and it grows by a factor of 1.5 times each day. In this If we graph of the function $h(x) = (0.01) \cdot (1.5)^x$, along with the linear function $f(x) = 10,000 + 1,000x$, we see that it takes many, many more days before we see a rapid growth. But given time to continue growing, the amount in this exponential option will eventually also outpace that in the linear option. When we say that exponential functions will eventually grow faster than linear functions, we are comparing their end behavior.



## Cool-down: Which One Gets There First? (*5 minutes*)

**Addressing:** NC.M1.F-LE.3

**Cool-down Guidance:** Press Pause
Take a few minutes prior to the start of Lessons 19 & 20 (modeling lessons) to discuss a few approaches to estimate values to compare the two functions.

### Cool-down

The function $f$ is given by $f(x) = 10x + 3$ and the function $g$ is given by $g(x) = 2^x$. For each question, show your reasoning.

1. Which function reaches 50 first?
2. Which function reaches 100 first?

**Student Reflection:**

How do you feel about the way you participated in class today? How might you want to improve your participation in future lessons?

### DO THE MATH

### INDIVIDUAL STUDENT DATA

### SUMMARY DATA

## NEXT STEPS

## TEACHER REFLECTION

**What went well in the lesson? What would you do differently next time? What happened today that will influence the planning of future lessons?**

**Reflect on your experience with the instructional routines in this lesson and across the curriculum. What routines are you most comfortable facilitating? Which have improved the learning for your students? Which do you want more support with?**

## Practice Problems

1. Functions $a$, $b$, $c$, $d$, $e$, and $f$ are given below. Classify each function as linear, exponential, or neither.

   a. $a(x) = 3x$

   b. $b(x) = 3^x$

   c. $c(x) = x^3$

   d. $d(x) = 9 + 3x$

   e. $e(x) = 9 \cdot 3^x$

   f. $f(x) = 9 \cdot 3x$

2. Here are four equations defining four different functions, $a$, $b$, $c$, and $d$. List them in order of increasing rate of change. That is, start with the one that grows the slowest and end with the one that grows the quickest.

   - $a(x) = 5x + 3$
   - $b(x) = 3x + 5$
   - $c(x) = x + 4$
   - $d(x) = 1 + 4x$

3. (*Technology required.*) Function $f$ is defined by $f(x) = 3x + 5$ and function $g$ is defined by $g(x) = (1.1)^x$.

| $x$ | $f(x)$ | $g(x)$ |
|---|---|---|
| 1 | | |
| 5 | | |
| 10 | | |
| 20 | | |

   a. Complete the table with values of $f(x)$ and $g(x)$. When necessary, round to 2 decimal places.

   b. Which function do you think grows faster? Explain your reasoning.

   c. Use technology to create graphs representing $f$ and $g$. What graphing window do you have to use to see the value of $x$ where $g$ becomes greater than $f$ for that $x$?

4. Functions $m$ and $n$ are given by $m(x) = (1.05)^x$ and $n(x) = \frac{5}{8}x$. As $x$ increases from 0:

   a. Which function reaches 30 first?

   b. Which function reaches 100 first?

5. The functions $f$ and $g$ are defined by $f(x) = 8x + 33$ and $g(x) = 2 \cdot (1.2)^x$.

   a. Which function eventually grows faster, $f$ or $g$? Explain how you know.

   b. Explain why the graphs of $f$ and $g$ meet for a positive value of $x$.

6. For each function rule, identify the percent increase or percent decrease.

   a. $f(x) = 11(1.34)^x$

   b. $g(x) = 86(0.82)^x$

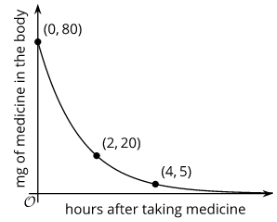
   (From Unit 6, Lesson 17)

7. The average price of a gallon of regular gasoline in 2016 was \$2.14. In 2017, the average price was \$2.42 a gallon—an increase of 13%.

   At that rate, what will the average price of gasoline be in 2020?

   (From Unit 6, Lesson 16)

8. The function $f$ represents the amount of a medicine, in mg, in a person's body $t$ hours after taking the medicine. Here is a graph of $f$.



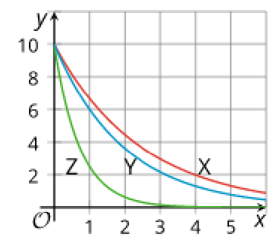

   a. How many mg of the medicine did the person take?

   b. Write an equation that defines $f$.

   c. After 7 hours, how many mg of medicine remain in the person's body?

   (From Unit 6, Lesson 15)

9. Here are the graphs of three functions. Which of these functions decays the most quickly? Which one decays the least quickly?




   (From Unit 6, Lesson 14)

10. A piece of paper is 0.004 inches thick.

    a. Explain why the thickness in inches, $t$, is a function of the number of times the paper is folded, $n$.

    b. Using function notation, represent the relationship between $t$ and $n$. That is, find a function $f$ so that $t = f(n)$.

    (From Unit 6, Lesson 11)

11. For each of the functions $f$, $g$, $h$, $p$, and $q$, the domain is $0 \leq x \leq 100$.

    For which functions is the average rate of change a good measure of how the function changes for this domain? Select **all** that apply.

    a. $f(x) = x + 2$
    b. $g(x) = 2^x$
    c. $h(x) = 111x - 23$
    d. $p(x) = 50,000 \cdot 3^x$
    e. $q(x) = 87.5$

    (From Unit 5)

# Lessons 19 & 20: Mathematical Modeling[1]

## PREPARATION

| Lesson Goals | Learning Targets |
|---|---|
| • Use mathematical tools to represent, interpret, analyze, generalize, communicate about, and make predictions about how real-world quantities vary in relation to each other.<br>• Use observational and experimental data to test, improve, and validate mathematical models. | • I can use mathematics to model real-world situations.<br>• I can test and improve mathematical models for accuracy in representing and predicting real things. |

### Lesson Narrative

Modeling is the link between the mathematics students learn in school and the problems they will face in college, career, and life. Time spent on modeling is crucial, as it prepares students to use math to handle technical subjects in their further studies, and problem solve and make decisions that adults regularly encounter in their lives.

For Lessons 19 and 20, there are six choices of modeling prompts: Modeling Prompts #1 and #2 are available in Unit 2; Modeling Prompts #3 and #4 are available in Unit 4; and Modeling Prompts #5 and #6 are provided here.

Remind students what modeling is and what is expected of them as a modeler using the following resources and guidance:

**Organizing Principles about Mathematical Modeling**

- The purpose of mathematical modeling in school mathematics courses is for students to understand that they can use math to better understand things in the world that interest them.
- Mathematical modeling is different from solving word problems. It often feels like initially there is not enough information to answer the question. There should be room to interpret the problem. There ought to be a range of acceptable assumptions and answers. Modeling requires genuine choices to be made by the modeler.
- It is expected that students have support from their teacher and classmates while modeling with mathematics. It is not a solitary activity. Assessment should focus on feedback that helps students improve their modeling skills.

**Things the Modeler Does When Modeling with Mathematics (NGA 2010)**

1. *Pose a problem* that can be explored with quantitative methods. Identify variables in the situation and select those that represent essential features.
2. *Formulate a model*: Create and select geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between variables.
3. *Compute*: Analyze these relationships and perform computations to draw conclusions.
4. *Interpret* the conclusions in terms of the original situation.
5. *Validate* the conclusions by comparing them with the situation. Iterate if necessary to improve the model.
6. *Report* the conclusions and the reasoning behind them.

It's important to recognize that in practice, these actions don't often happen in a nice, neat order.

[1] Adapted from IM 9–12 Math Algebra 1 Modeling Prompts https://curriculum.illustrativemathematics.org/HS/index.html, 

## Preparing for a Modeling Prompt

### Ideas for Setting Up an Environment Conducive to Modeling

- Provide plenty of blank whiteboard or chalkboard space for groups to work together comfortably. “Vertical non-permanent surfaces” are most conducive to productive collaborative work. “Vertical” means on a vertical wall, which is better than horizontally on a tabletop, and “non-permanent” means something like a dry erase board, which is better than something like chart paper (Liljedahl 2016).
- Ensure that students have easy access to any tools that might be useful for the task. These might include:
  - supply table containing geometry tools, calculators, scratch paper, graph paper, dry erase markers (ideally a different color for each group member), post-its
  - electronic devices to access digital tools (like graphing technology, dynamic geometry software, or statistical technology)
- Think about how to help students manage the time that is available to work on the task. For example:
  - Display a countdown timer for intermittent points in the lesson when groups are asked to summarize their progress.
  - Decide what time to ask groups to transition to writing down their findings in a somewhat organized way (perhaps 15 minutes before the end of the class).

### How to Prepare and Conduct the Modeling Lesson

- Decide which version of the prompt students will receive, based on the lift-analysis, timing, and access to data.
- Have data ready to share if planning to give it when students ask.
- Decide if students will be offered a template for organizing modeling work.
- Decide to what extent students are expected to iterate and refine their model. The amount of time available can influence how much time students have to refine their model. If time is short, students may not engage as much in that part of the modeling cycle. WIth more time, it is more reasonable to expect students to iterate and refine their model once or even several times.
- Decide how students will report their results. Again, if time is short, this may be a rough visual display on a whiteboard. If more time is available, students might create a more formal report, slideshow, blog post, poster, mockup of an artifact like a letter to a specific audience, smartphone app, menu, or set of policies for a government entity to consider. One way to scaffold this work is to ask students to turn in a certain number of presentation slides: one that states the assumptions made, one that describes the model, and one or more slides with their conclusions or recommendations.
- Develop task-specific “look-fors” for each dimension of the provided rubric. What do you anticipate and hope to see in student work?

### Organizing Students Into Teams or Groups

- Mathematical modeling is not a solitary activity. It works best when students have support from each other and their teacher.
- Working with a team can make it possible to complete the work in a finite amount of class time. For example, the team may decide it wants to vary one element of the prompt and compute the output for each variation. What would be many tedious calculations for one person could be only a few calculations for each team member.
- The members of good modeling groups bring a diverse set of skills and points of view. Create and share a Multiple Abilities List with students.
- Scramble the members of modeling teams often, so that students have opportunities to play different roles.

### Ways to Support Students While They Work on a Modeling Prompt

- Coach students on ways to organize their work.
- Provide a template to help students organize their thinking. Over time, some groups may transition away from needing to use a template.
- Engage students in the *Three Reads* instructional routine to ensure comprehension of the prompt.
- Remind students of the variety of tools that are available to them.
- If students get stuck or run out of ideas, help move them forward with a question that prompts them to focus on a specific part of the modeling cycle. For example:
  - “What quantities are important? Which ones change and which ones stay the same?”
  - “What information do you know? What information would it be nice to know? How could you get that information? What reasonable assumption could you make?”
  - “What pictures, diagrams, graphs, or equations might help people understand the relationships between the quantities?”
  - “How are you describing the situation mathematically? Where does your solution come from?”
  - “Under what conditions does your model work? When might it not work?”
  - “How could you make your model better? How could you make your model more useful under more conditions?”
  - “What parts of your solution might be confusing to someone reading it? How could you make it more clear?”

**How to Interpret the Provided Lift Analysis of a Modeling Prompt**

For most mathematical modeling prompts, different versions are provided. Each version is analyzed along five impactful dimensions that vary the demands on the modeler (OECD 2013). Each of the attributes of a modeling problem is scored on a scale from 0–2. A lower score indicates a prompt with a “lighter lift” for students and teachers: students are engaging in less open, less authentic mathematical modeling. A higher score indicates a prompt with a “heavier lift” for students and teachers: students are engaging in more open, more authentic mathematical modeling. This matrix shows the attributes that are part of our analysis of each mathematical modeling prompt. Though not all the attributes have the same impact on what teachers and students do, for the sake of simplicity, they are all weighted the same when they are averaged.

| **Attribute** | | DQ<br>Defining the Question | QI<br>Quantities of Interest | SD<br>Source of Data | AD<br>Amount of Data given | M<br>The Model |
|---|---|---|---|---|---|---|
| **Lift** | Light lift (0) | Well-posed question | Key variables declared | Data are provided | Modeler is given all the information they need and no more | Model is given in the form of a mathematical representation |
| | Medium lift (1) | Elements of ambiguity; prompt might suggest ways assumptions could be made | Key variables suggested | Modelers are told what measurements to take or data to look up | Some extra information is given and modeler must decide what is important; or, not enough information is given and modeler must ask for it before teacher provides it | Modeler must sift through lots of given information and decide what is important; or, not enough information is given and modeler must make assumptions, look it up, or take measurements |
| | Heavy lift (2) | Freedom to specify and simplify the prompt; modeler must state assumptions | Key variables not evident | Modelers must decide what measurements to take or data to look up | Modeler must sift through lots of given information and decide what is important; or, not enough information is given and modeler must make assumptions, look it up, or take measurements | Careful thought about quantities and relationships or additional work (like constructing a scatter plot or drawing geometric examples) is required to identify type of model to use |

Each version of a mathematical modeling prompt is accompanied by an analysis chart that looks like this sample:

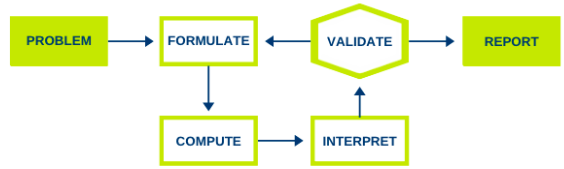
| **Attribute** | DQ | QI | SD | AD | M | Avg |
|---|---|---|---|---|---|---|
| **Lift** | 0 | 1 | 0 | 0 | 2 | 0.6 |

There are other features of a mathematical modeling prompt that could be varied. In the interest of not making things too complex, there are only five dimensions included in the lift analysis. However, a prompt could be additionally modified on one of these dimensions:

- whether the scenario is posed with words, a highly-structured image or video, or real-world artifacts like articles or authentic diagrams
- presenting example for student to explore before they are expected to engage with the prompt, versus the prompt suggesting that the modeler generate examples or expecting the modeler to generate examples on their own
- whether the prompt makes decisions about units of measure or expects the modeler to reconcile units of measure or employ dimensional thinking
- whether a pre-made digital or analog tool is provided, instructions given for using a particular tool, use of a particular tool is suggested, or modelers simply have access to familiar tools but are not prompted to use them
- whether a mathematical representation is given, suggested, or modelers have the freedom to select and create representations of their own choosing

## Advice on Modeling

These are some steps that successful modelers often take and questions that they ask themselves. You don't necessarily have to do all of these steps, or do them in order. Only do the parts that you think will help you make progress.




**Understand the Question**
Think about what the question means before you start making a strategy to answer it. Are there words you want to look up? Does the scenario make sense? Is there anything you want to get clearer on before you start? Ask your classmates or teacher if you need to.

**Refine the Question**
If necessary, rewrite the question you are trying to answer so that it is more specific.

**Estimate a Reasonable Answer**
If you don't have enough information to decide what's reasonable, try to come up with an answer that would be too low, and an answer that would be too high.

**Identify Unknowns**
- What are the meaningful quantities in this situation? Write them down.
- What information would be useful to know? In order to get that information, you could: look it up, take a measurement, or make an assumption.

**Gather Information**
Write down any of the unknown information that you find. As you work, organize your information in a way that makes sense to you.

**Experiment!**
Try different ideas to make progress toward answering your question. If you are stuck, think about:
- Helpful ways to organize the information you have or organize your work
- Questions you *can* answer using the information you have
- Ways to represent mathematical relationships or sets of data (tables, equations, scatter plots, graphs, statistical plots)
- Tools that are available for representing mathematics, both digital and analog

**Check Your Reasoning**
Do you have a first answer to your question? Great! See if it's reasonable.
- Make sure you can explain what the answer means in terms of the original problem.
- Check your precision: Is your answer overly precise (do you really need all those decimal places)? Not precise enough (were you overly aggressive with your rounding)?

**Use and Improve Your Model**
- Did you make assumptions or measurements? How can you express your model more generally, so that it would work for a range of numbers instead of the specific numbers you used?
- What are the limitations of your model? That is, what are some ways it is not realistic? Does it only work for certain inputs but not others? Are there any meaningful inputs affecting the outcome that are not accounted for? If possible, improve your model to take these into account.
- What are the implications of your model? That is, what should people or organizations do differently or smarter as a result of what your model shows? What would be effective ways to communicate with them?
- What are the areas for further research? That is, what new things are you wondering about that could be investigated, by you or someone else?

## Modeling Rubric

| Skill | Score | | | Notes or Comments |
|---|---|---|---|---|
| | Proficient | Developing | Needs Revisiting | |
| 1. Decide What to Model | • Assumptions made are clearly identified and justified. Resulting limitations are stated when appropriate.<br>• Variables of interest are clearly identified and chosen wisely, and appropriate units of measure are used. | • Assumptions are noted but lacking in justification or difficult to find.<br>• Variables of interest are noted, but may lack justification, be difficult to find, or not be measured with appropriate units. | • No assumptions are stated.<br>• No variables are defined. | |
| | To improve at this skill, you could:<br>• Ask questions about the situation to understand it better<br>• Check the assumptions you're making to see if they're reasonable (Try asking a friend, or imagining that you're a person involved in the scenario. Would those assumptions make sense to you?)<br>• Double-check the variables you've identified: Are there other quantities in the situation that could vary? Is there something you've identified as a variable that is actually fixed or determined? (Remember that more abstract things like time and speed are also quantities.) | | | |
| 2. Formulate a Mathematical Model | • An appropriate model is chosen and represented clearly.<br>• Diagrams, graphs, etc. are clear and appropriately labeled. | Parts of the model are unclear, incomplete, or contain mistakes. | No model is presented, or the presentation contains significant errors. | |
| | To improve at this skill, you could:<br>• Check your model more carefully to make sure it really fits well<br>• Consider a wider variety of possible models, to find one that fits the situation better<br>• Think about the situation more deeply before trying to find a model<br>• Convince a skeptic: Pretend that you think your model is inadequate, or ask a friend to pretend to be skeptical of it. What would a skeptic find wrong with your model? Try to fix those things, or explain why they're not actually problems. | | | |

| Skill | Score | | | Notes or Comments |
|---|---|---|---|---|
| | Proficient | Developing | Needs Revisiting | |
| 3. Use Your Model to Reach a Conclusion | • Solution is relevant to the original problem.<br>• Reader can easily understand the reasoning leading to the solution.<br>• Relevant details are included like units of measure. | Solution is not well-aligned to the original problem, or aspects of the solution are difficult to understand or incomplete. | No solution is provided. | |
| | To improve at this skill, you could:<br>• Double-check your calculations: Show them to someone else to see if they agree, or take a break and look at your calculations again later<br>• Make sure your calculations are justified by your model: Ask yourself how you decided what to calculate, and see if your reasoning matches up with your model<br>• Think more deeply about what your conclusions mean in the original scenario: Imagine you're a person involved in the scenario, or explain your conclusions to someone else and see if they have questions | | | |
| 4. Refine and Share Your Model | • The model's implications are clearly stated.<br>• The limitations of the model and solution are addressed. | The limitations of the model and solution are addressed but lacking in depth or ignoring key components. | No interpretation of model and solution is provided. | |
| | To improve at this skill, you could:<br>• Think more creatively about what your conclusions mean: Ask yourself "If I was involved in this situation, what would I understand better because of these conclusions? What would I want to do next?"<br>• Be skeptical of your model: What don't you like about it, and what can you do to fix those things?<br>• Explain your model to someone else: Tell them how it works and why it's good. If you're not sure how it works or why it's good, you might need to change it. | | | |

**What aspects of modeling are your students best at doing? What aspects of modeling do students need to strengthen? How will you honor those strengths and areas in need of growth?**

## Agenda, Materials, and Preparation

- **Modeling Prompts #1 & #2: See Math 1 Unit 2 materials**
- **Modeling Prompts #3 & #4: See Math 1 Unit 4 materials**
- **Modeling Prompt #5: Giving Bonuses**
  - Modeling Prompt #5 (print 1 copy per student)
  - Modeling Rubric (print 1 copy per student)
- **Modeling Prompt #6: Shoulder to Shoulder**
  - Modeling Prompt #6 (print 1 copy per student)
  - Modeling Rubric (print 1 copy per student)
  - Packed crowd slides: https://bit.ly/PackedCrowdSlides (print or display for students to reference)
  - Location data sheet for prompt 6B: https://bit.ly/LocationData6B
  - "7 Billion" video from National Geographic: https://bit.ly/NatGeo7billion

## LESSON

### Modeling Prompt #5: Giving Bonuses

In this modeling prompt, students will work on determining a “fair” way to distribute bonuses to five employees at a company who worked on a specific project. There are two versions of this prompt: 5A and 5B. In 5A, students are not given information on the five employees so will need to make assumptions upon which to base their recommendations. In 5B, students are given the job description, salary, years of experience, and hours worked on the project. Determine, in advance, which Modeling Prompt (5A or 5B) students will receive, based on the lift-analysis, timing, and access to data.

**Student Task Statement 5A Lift Analysis**

| **Attribute** | DQ | QI | SD | AD | M | Avg |
|---|---|---|---|---|---|---|
| **Lift** | 1 | 2 | 2 | 2 | 2 | 1.8 |

**Student Task Statement 5B Lift Analysis**

| **Attribute** | DQ | QI | SD | AD | M | Avg |
|---|---|---|---|---|---|---|
| **Lift** | 1 | 1 | 0 | 1 | 1 | 0.8 |

**Step 1** (*Optional;* review modeling materials as necessary)

- Display and pass out the Advice on Modeling and Modeling Rubric handouts.
- Facilitate a discussion around modeling. Share some of the following ideas:
  - Modeling prompts are often expressed in words, but unlike word problems, modeling prompts challenge the modeler to make reasonable assumptions, decide what information is important, ask or research for more information if needed, think creatively within constraints, and consider the implications of the model.
  - The process of modeling is cyclical, and it does not end by producing a “correct answer.”
  - A mathematical model expresses a simplified relationship in the real world; models can be rough but still useful.
  - Models can often be refined to represent the real-world relationship more accurately.
  - Responses to modeling prompts can vary widely, but they often contain certain pieces: assumptions, calculations, a mathematical model (stated with an equation or equations, with a graph, with a geometric diagram, or in words), conclusions, and generalizations.
- Provide time for students to ask clarifying questions. Students should have a good-enough understanding of the rubric, however, deep understanding of the rubric is not needed at this time.

**Step 2**

- Pass out the pre-determined appropriate blackline master Modeling Prompt #5 (5A or 5B) and Modeling Rubric if it has not already been distributed.
- Students can be arranged in groups in advance, they can choose groups, or groups can be determined by using visibly random grouping.

**Modeling Prompt 5A**

A project at a large company was very successful, and the company made more money than expected as a result. Your boss has given you the task of coming up with different methods to distribute bonuses to the five employees that directly worked on the project. There is a total of $8,000 available to distribute.

The method for distributing the money will be shared with the entire company, so it is important that the employees feel the distribution is fair.

1. Make a proposal with at least two different methods for your boss to choose from. Describe the advantages and disadvantages of each method. Then give your recommendation and provide an argument in its support.
2. For each of the methods you propose, which of the five employees is most likely to complain about the method being unfair? How would you justify the method to this employee?

## Modeling Prompt 5B

A project in a large company was very successful, and the company made more money than expected as a result. Your boss has given you the task of coming up with different methods to distribute bonuses to the five employees that directly worked on the project. There is a total of $8,000 available to distribute.

Here is some information about the employees:

| Employee | Job description | Hours working on project (per week) | Annual salary | Job experience |
|---|---|---|---|---|
| A | Receptionist | 40 | $30,000 | 1 year |
| B | Administrative coordinator | 30 | $30,000 | 5 years |
| C | Manager | 40 | $80,000 | 3 years |
| D | Sales representative | 40 | $50,000 | 10 years |
| E | Sales representative | 20 | $20,000 | 2 years |

1. Make a proposal with at least two different methods for your boss to choose from. Outline the advantages and disadvantages of each method. Then give your recommendation and support your argument.
2. For each of the methods you propose, which of the five employees is most likely to complain about the method being unfair? How would you justify the method to this employee?

### Step 3

- As students are considering the situation, prompt them with the following questions, as necessary:
    - "What are some variables you should consider?"
    - "Can you think of mathematically different methods to distribute the money?"
    - "What information would be helpful to know to compute the bonus for each employee?"
- Tell students that they can make up the information they need to compute specific bonuses for two different methods and to apply them to the situation. In the proposal, students should compute the specific dollar amount each employee receives as a bonus.
- If there is time, encourage students to try generalizing any values they invented to use variables instead.

### Step 4

- Remind students that modeling is a cycle, and that they should evaluate their own models and then refine them, as necessary.
- After sufficient work time, each group or pair should share their solutions with the class. Students could share by presenting to the class, doing a gallery walk, creating a slide deck, uploading a scan or photo of their work to a shared online space, or by any method that works best for the class.

### Step 5

- Provide students time to reflect on their experience with this modeling prompt in their Student Workbook.

## Modeling Prompt #6: Shoulder to Shoulder[2]

**Instructional Routines:** Notice and Wonder; Round Robin

Often there are debates about the capacity of our world and if we have enough space in the world for the human population to grow. To better understand what this debate is about, students begin exploring how much area the current human population would take if we all stood shoulder to shoulder. Students are often surprised to learn that the entire human population fits in a relatively small area. Of course, we do not live under these conditions, and when it comes to the capacity of our world, there are other issues to consider beyond the area we cover. This modeling prompt sets the stage for students to consider other factors that are integral to the debate about the human carrying capacity of our earth (e.g., resources, population density).

This prompt launches with students seeing images of packed crowds and engaging in the *Notice and Wonder* routine to surface initial observations and wonderings about the images of people crowded together.

Students begin the process of estimating, determining what information they will need, and making working assumptions. After individually brainstorming how they will approach the question, students continue to tackle the task in pairs and identify their assumptions, choose their strategies, do the necessary calculations, and reach their conclusions. Ultimately they will create a product of their results such as a poster to share through a gallery walk.

In this modeling prompt, students will work on using areas to grasp the magnitude of very large numbers within the context of the world's population. There are two versions of this prompt: 6A and 6B. In 6A, students are asked to define and research the information they need in order to complete the prompt. . In 6B, students are provided with some information that reduces the number of unknown variables in the model they're building. Determine, in advance, which Modeling Prompt (6A or 6B) students will receive, based on the lift-analysis, timing, and access to data.

**Student Task Statement 6A Lift Analysis**

| **Attribute** | DQ | QI | SD | AD | M | Avg |
|---|---|---|---|---|---|---|
| **Lift** | 1 | 2 | 1 | 1 | 1 | 1.2 |

**Student Task Statement 6B Lift Analysis**

| **Attribute** | DQ | QI | SD | AD | M | Avg |
|---|---|---|---|---|---|---|
| **Lift** | 0 | 1 | 1 | 1 | 1 | 0.8 |

### **Step 1** (*Optional;* review modeling materials as necessary)

- Display and pass out the Advice on Modeling and Modeling Rubric handouts.
- Facilitate a discussion around modeling. Share some of the following ideas:
  - Modeling prompts are often expressed in words, but unlike word problems, modeling prompts challenge the modeler to make reasonable assumptions, decide what information is important, ask or research for more information if needed, think creatively within constraints, and consider the implications of the model.
  - The process of modeling is cyclical, and it does not end by producing a "correct answer."
  - A mathematical model expresses a simplified relationship in the real world; models can be rough but still useful.
  - Models can often be refined to represent the real-world relationship more accurately.
  - Responses to modeling prompts can vary widely, but they often contain certain pieces: assumptions, calculations, a mathematical model (stated with an equation or equations, with a graph, with a geometric diagram, or in words), conclusions, and generalizations.
- Provide time for students to ask clarifying questions. Students should have a good-enough understanding of the rubric, however, deep understanding of the rubric is not needed at this time.

[2] Developed with MathAction ©2021. Materials are intended for classroom use only; distribution or use beyond individual classrooms is prohibited. For more problem based lessons visit www.mathaction.org.

### Step 2

- Show students four or five slides of a packed crowd: https://bit.ly/PackedCrowdSlides.
- Using the *Notice and Wonder* routine, ask students what they notice or wonder about these images. Each image has a short description on the slide that may help to respond briefly to students' wonders.
- Invite students to share their noticings and wonders.
- Ask students to guess where the entire school population might fit if they were huddled together, shoulder to shoulder, without any extra space. Record a few of their guesses.

### Step 3

- Pass out the pre-determined appropriate blackline master Modeling Prompt #6 (6A or 6B) and Modeling Rubric if it has not already been distributed.
- After students find the area covered by the entire world human population standing shoulder to shoulder, students find a geographical location (e.g., a city, country) that could theoretically host the entire human population without much space left over.
  - Students using Modeling Prompt 6A can use the internet to research an appropriate geographical location.
  - Students completing Modeling Prompt 6B should be provided with a copy of the location data sheet: https://bit.ly/LocationData6B.
- Students can be arranged in groups in advance, they can choose groups, or groups can be determined by using visibly random grouping.

### Modeling Prompt 6A

If all the people in the world huddled together shoulder to shoulder, without any extra space, how much area would we all cover? What geographical location (e.g., a city, country, continent) could theoretically host the entire human population without much space left over?

### Modeling Prompt 6B

If all 8 billion (8,000,000,000) people in the world huddled together shoulder to shoulder, without any extra space, how much area would we all cover? What geographical location (e.g., a city, country, continent) could theoretically host the entire human population without much space left over?

### Step 4

- Ask students to make a guess at a location that the whole world could just fit into and write it in their Student Workbook.
- To help students further examine their conjectures/guesses, ask, “What would you need to know in order to answer this question?”
- Students independently brainstorm how they will approach the problem.
- If appropriate, ask students to consider the following questions:
  - “What assumptions do you need to make in order to solve this problem?”
  - “What strategies do you want to try as you tackle this problem?”

Using the *Round Robin* routine, ask students to compare and contrast their strategies and ideas with their small groups and then choose a method to use to complete the prompt.

### Step 5

- Remind students that modeling is a cycle, and that they should evaluate their own models and then refine them, as necessary.
- After sufficient work time, each group should share their solutions with the class. Students could share by presenting to the class, creating a poster and doing a gallery walk, creating a slidedeck, uploading a scan or photo of their work to a shared online space, or by any method that works best for the class.

### Step 6

- Provide students time to reflect on their experience with this modeling prompt in their Student Workbook.

### Step 7

- Debrief this modeling activity with an opportunity to reflect on the prompt, discuss important mathematical moments, and solidify and extend the learning.
- Connect students' final conclusion with their initial assumptions and highlight how the varied answers are the result of different assumptions. Facilitate a discussion by asking students the following questions:
  - "How does the location compare with where you initially predicted everyone could fit?"
  - "What do you notice about our initial predictions?"
  - "What do you understand now?"
  - "Do you think the amount of land on Earth limits how large the world's population can get? Why or why not?"
  - "If the amount of land on Earth is not a limiting factor on population growth, then what are limiting factors?"
- Consider having students complete the reflection, "I used to think______, and now I know __________."
- Consider watching the "7 Billion" video from National Geographic at the conclusion of the lesson: https://bit.ly/NatGeo7billion. The video is from 2011 so the numbers are different, but the concepts and compelling questions remain.

## TEACHER REFLECTION

**Which modeling prompt did you choose for this lesson? What version of the prompt did students receive? How did you make these decisions? Would you make a different choice next time you facilitate this lesson?**

# Lesson 21: Post-Test Activities

## PREPARATION

| Lesson Goals | Learning Targets |
|---|---|
| • Choose and write a linear or exponential function to model real-world data.<br>• Determine and explain (in writing) how well a function models the given data.<br>• Use given population data to calculate or estimate growth rates and make predictions. | • I can determine whether to use a linear function or an exponential function to model real-world data.<br>• I can determine how well a chosen model fits the given information. |

### Lesson Narrative

This lesson, which should occur after the Unit 6 End-of-Unit Assessment, allows for students to reflect on the unit, share feedback, conference with the teacher, and engage in activities that support the work of the upcoming unit.

Gathering student feedback is a powerful and strategic way to learn about students and improve instructional practices. It also creates student and family buy-in and centers students as decision makers and problem solvers in their own learning.

**What do you hope to learn about your students during this lesson?**

### Agenda, Materials, and Preparation

- **Activity 1** (*20 minutes*)
  - End-of-Unit 6 Student Survey (print 1 copy per student)
- **Activity 2** (*25 minutes*)
  - Technology is required for this activity: Acquire devices that can access Desmos (recommended) or other graphing technology. It is ideal if each student has their own device.
  - Desmos Regressions Steps resource (print 1 copy per student as needed or provide link: https://bit.ly/RegressionResource)
  - World Population Clock link: https://bit.ly/PopulationWorld

## LESSON

### Activity 1: End-of-Unit 6 Student Survey (*20 minutes*)

The End-of-Unit 6 Student Survey is a critical opportunity for teachers to gather low-stakes, non-evaluative feedback to support equity and instructional pedagogy. The survey is also highly beneficial for students as it is designed to encourage self-awareness, self-management, social awareness, relationship skills, and responsible decision making. Provide students a chance to quietly and independently complete this survey after they complete their testing.

## One-on-One Conferences

Conducting one-on-one conferences with students, using the surveys as a data point, is encouraged. These conferences can be done as students complete their surveys and are engaging in Activity 2. Potential conference topics include:

- student responses to the daily student reflections
- student response to the end-of-unit student survey (as students finish them)
- executive functioning skills
- student learning contracts
- goal setting and self-evaluation

## Activity 2: Population Predictions (*25 minutes*)

**Instructional Routine:** Notice and Wonder

In this activity, students use the main function types they have studied thus far in the course (linear and exponential) to model different populations. Students begin with using the *Notice and Wonder* routine after data for three city populations are given. Then, students are asked to produce a linear or exponential model for each (if appropriate) and then make predictions for populations at other dates. The cities have been chosen so that one is well modeled by an exponential model, another by a linear model, and the third by neither. Students then examine world population with a task that is more open-ended, with only limited data provided, likely requiring students to gather more data. In addition, the world population is *not* consistently well modeled by a linear or exponential function, but for certain periods of time, exponential and/or linear functions can be appropriate (in particular, in recent years the growth has been strikingly linear).

Students should work on this task in groups of three, with each individual student creating a separate graph for one of the three cities with the population data given. Students should engage in discussions for each question asking about the appropriateness of different types of models, with each student using sentence frames for sharing their opinions and to agree or disagree with other group members. Students should have access to the Desmos Regression Steps resource if needed in order to find model linear and exponential functions throughout the activity: https://bit.ly/RegressionResource.

### Student Task Statement

Here are population data for three cities at different times between 1950 and 2000. The following questions and prompts will help you to decide what the data tells us, if anything, about the current population in the cities or what the population will be in 2050?[1]

| City | 1950 | 1960 | 1970 | 1980 | 1990 | 2000 |
|---|---|---|---|---|---|---|
| Paris | 6,300,000 | 7,400,000 | 8,200,000 | 8,700,000 | 9,300,000 | 9,700,000 |
| Austin | 132,000 | 187,000 | 254,000 | 346,000 | 466,000 | 657,000 |
| Chicago | 3,600,000 | 3,550,000 | 3,400,000 | 3,000,000 | 2,800,000 | 2,900,000 |

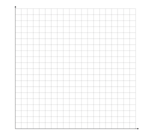
1. What do you notice? What do you wonder? Take a minute to think individually and then share with your group.
2. Each student in the group should select a different city. How would you describe the population change in each city during this time period? Write one to two sentences, then discuss with your group.
3. Use technology to graph the population data for each city, with each student creating a graph for their selected city and sharing with the group.

[1] Adapted from IM 9–12 Math Algebra 1, Unit 5, Lesson 21 https://curriculum.illustrativemathematics.org/HS/teachers/index.html, copyright 2019 by Illustrative Mathematics. Licensed under the Creative Commons Attribution 4.0 license https://creativecommons.org/licenses/by/4.0/.

4. What kind of model (linear, exponential, both, or neither) do you think is appropriate for each city population? Each group member should make a choice for their selected city and explain their reasoning to the group. Use the sentence frames to engage in discussion:

    - A ____ model is appropriate to model the population data for ______ because _____.
    - I agree a ____ model is appropriate to model the population data for ____ because ___ and also because ___.
    - I disagree that ___model is appropriate to model the population data for ___ because ___.

5. Working together with your group members, for each population that you agreed can be modeled by a linear and or exponential function:



    a. Write an equation for the function(s).

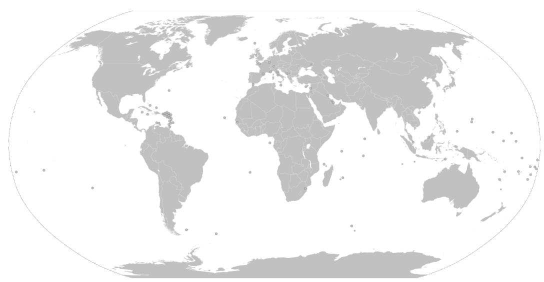
    b. Graph the function(s) and the data on the same coordinate plane.

6. Working together with your group members, compare the graphs of your functions with the actual population data to determine how well the models fit the data.

    a. Use your models to predict the population in each city in 2010, the current year, and 2050 (three predictions).

    b. Do you think that these predictions are (or will be) accurate? Explain your reasoning.

Now you will explore world population data. If you would like to include additional data points, use the first two columns of the historical data table found at: https://bit.ly/PopulationWorld. All of these questions should be completed collaboratively with your group.



(Image[2])

| **Year** | 1804 | 1927 | 1960 | 1974 | 1987 | 1999 | 2011 | 2020 |
|---|---|---|---|---|---|---|---|---|
| **World population in billions** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 7.8 |

7. Use technology to graph the world population data. Make sure you adjust the window to fit all of your data.

8. Take a minute to think individually about whether or not a linear function would be appropriate for modeling the world population growth over the last 200 years. Use the sentence frames to engage in discussion:

    - A linear model (would/would not) be appropriate because ___.
    - I agree a linear model (is/is not) appropriate to model the world population data for the last 200 years because ___ and also because ___.
    - I disagree that a linear model (is/is not) appropriate to model the world population data for because ___.

9. If you agree that a linear model is appropriate for the last 200 years, find a linear model using technology.

10. Take a minute to think individually about whether or not an exponential function would be appropriate for modeling the world population growth over the last 200 years. Use the sentence frames to engage in discussion:

    - An exponential model (would/would not) be appropriate because ___.
    - I agree an exponential model (is/is not) appropriate to model the world population data for the last 200 years because ___ and also because ___.
    - I disagree that an exponential model (is/is not) appropriate to model the world population data for because ___.

[2] https://commons.wikimedia.org/wiki/Maps_of_the_world#/media/File:BlankMap-World-with-Circles.svg. 

11. If you agree that an exponential model is appropriate for the last 200 years, find an exponential model using technology.

12. If the growth trend continues, use one or both of your models to predict what the world population will be in 2050. Do you think the prediction(s) are reasonable? Explain your reasoning.

13. Do you think there is a limit to how long the model will provide accurate predictions? Why or why not?

## TEACHER REFLECTION

**As you finish up this unit, reflect on the norms and activities that have supported each student in learning math. List ways you have seen each student grow as a young mathematician throughout this work.**

**List ways you have seen yourself grow as a teacher this year.**

**What will you continue to do and what will you improve upon in Unit 7?**